10.

CONTINUOUS PROBABILITY DISTRIBUTIONS

In this chapter you will expand the work you have done on discrete probability distributions in Year 11. You will study continuous probability distributions, including the normal distribution.

CHAPTER OUTLINE

10.01 Probability density functions
10.02 Calculating probabilities
10.03 Cumulative distribution function
10.04 Quantiles
10.05 Normal distribution
10.06 Empirical rule
10.07 *z*-scores
10.08 Applications of the normal distribution



IN THIS CHAPTER YOU WILL:

- recognise continuous random variables
- understand the properties of a probability density function (PDF)
- find cumulative distribution functions (CDF)
- find probabilities of continuous data
- calculate measures of central tendency and spread for continuous probability distributions
- recognise the normal distribution and identify its properties
- calculate probabilities and quantiles for normal distributions
- understand the standard normal distribution and *z*-scores
- apply the normal distribution to solving practical problems



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continuous random variable: A random variable that can have any value along a continuum, for example, the height of a basketball player.
cumulative distribution function: A function $F(x)$ for the probability $P(X \leq x)$.
empirical rule: The percentage probabilities (68%, 95%, 99.7%) that normally-distributed scores will lie within 1, 2, and 3 standard deviations, respectively, from the mean.
normal distribution: A continuous probability distribution in which the mean, mode and median are at the centre of a symmetrical bell-shaped graph.
probability density function: A function of a continuous random variable whose integral gives the probability $P(X \leq x)$.
random variable: A variable whose values are based on a chance experiment; for example, the number of road accidents in an hour
uniform probability distribution: A probability distribution in which every outcome has the same probability
***z*-score:** Measures how many standard deviations above or below the mean a score is.

Probability density functions

WS Probability density function

10.01 Probability density functions

EXAMPLE 1

This table gives the results of a survey of different times that runners take to complete a race.

Time (min)	Frequency
0–<4	6
4–<8	8
8–<12	11
12–<16	4
16–<20	2
20–<24	1

a Add a column of relative frequencies.
b Sketch a frequency histogram for the relative frequencies.
c Estimate each probability:

i $P(X < 12)$ **ii** $P(X \geq 16)$ **iii** $P(4 \leq X < 8)$

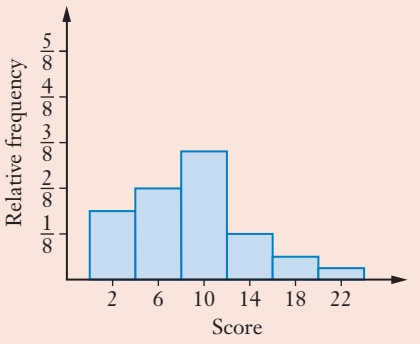
Solution

a

Time (min)	Frequency	Relative frequency
0–<4	6	$\frac{6}{32} = \frac{3}{16}$
4–<8	8	$\frac{8}{32} = \frac{1}{4}$
8–<12	11	$\frac{11}{32}$
12–<16	4	$\frac{4}{32} = \frac{1}{8}$
16–<20	2	$\frac{2}{32} = \frac{1}{16}$
20–<24	1	$\frac{1}{32}$

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b



c **i** $P(X < 12) = \frac{6}{32} + \frac{8}{32} + \frac{11}{32}$

$= \frac{25}{32}$

ii $P(X \geq 16) = \frac{2}{32} + \frac{1}{32}$

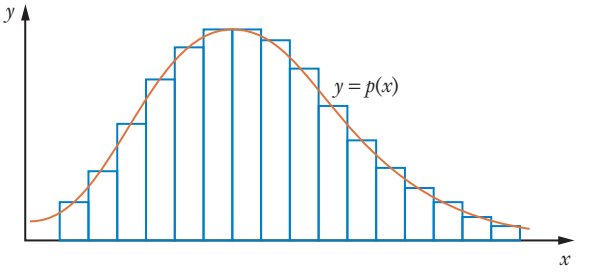
$= \frac{3}{32}$

iii $P(4 \leq X < 8) = \frac{1}{4}$

The times in the above example are values of a **continuous random variable**, but sorted into groups. While we can estimate probabilities using relative frequency, we use other methods when dealing with continuous data.

Continuous probability distributions

With continuous data we can't really draw a histogram as we did in Example 1 or there would be an 'infinite' number of columns with 'zero' widths. Instead, the probability distribution is a continuous curve.



A **continuous probability distribution** is represented by a function $P(X = x)$ or $p(x)$ called a **probability density function** (PDF) where X is the random variable. As with discrete probability distributions, the sum of all probabilities must be 1.

With a continuous probability distribution, we cannot calculate the probability for a single outcome, so $P(X = x) = 0$. Instead, we can only calculate the probability for a range of values such as $P(4 \leq X < 8)$.

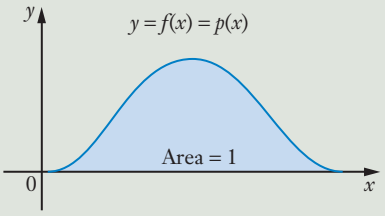
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Area under a probability density function

The area under a probability density function is 1.

$$\int_{-\infty}^{\infty} f(x)\,dx = 1$$

where $f(x) \geq 0$ (since $0 \leq p(x) \leq 1$)



EXAMPLE 2

a A function is given by $f(x) = \begin{cases} \dfrac{3x^2}{26} & \text{for } 1 \leq x \leq 3 \\ 0 & \text{for all other } x \end{cases}$.

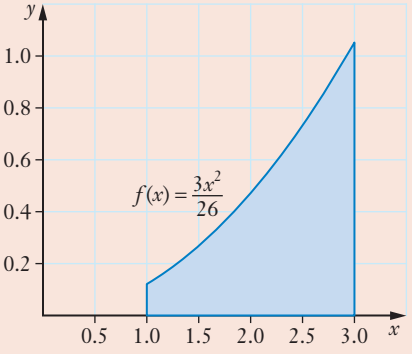
Show that it is a continuous probability distribution.

b A function is given by $f(x) = ax^2$ defined for the domain $[0, 5]$.
Find the value of a for which this is a probability density function.

Solution

a For a continuous probability distribution, the area under the curve must be 1.

Drawing the graph, notice that the area will be 0 for all x values outside $1 \leq x \leq 3$.



$$\text{Area} = \int_1^3 \frac{3x^2}{26}\,dx$$

$$= \frac{1}{26}\int_1^3 3x^2\,dx$$

$$= \frac{1}{26}\left[x^3\right]_1^3$$

$$= \frac{1}{26}[3^3 - 1^3]$$

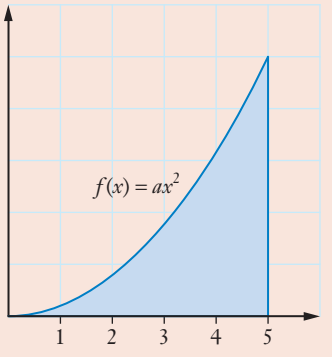
$$= \frac{1}{26}[26]$$

$$= 1$$

So $f(x)$ is a continuous probability distribution.

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b Drawing the graph gives a parabola in the domain [0, 5].



A PDF has area 1.

So $\int_0^5 ax^2\,dx = 1$

$$\int_0^5 ax^2\,dx = a\int_0^5 x^2\,dx$$

$$= a\left[\frac{x^3}{3}\right]_0^5$$

$$= a\left[\frac{5^3}{3} - \frac{0^3}{3}\right]$$

$$= a\left[\frac{125}{3}\right]$$

$$= \frac{125a}{3}$$

So $\frac{125a}{3} = 1$

$$a = \frac{3}{125}$$

Exercise 10.01 Probability density functions

1 State whether each random variable is discrete or continuous.

a The size of T-shirts worn by people

b The speed of cars as they pass a certain point

c The volume of water in a dam

d The number of seats on an aeroplane

e The weight of babies born in August

2 Which of the following functions describe continuous probability distributions?

a $f(x) = 0.2$ in the domain $[1, 6]$

b $f(x) = \frac{x}{12}$ in the domain $[0, 6]$

c $f(x) = \begin{cases} \frac{x^3}{324} & \text{for } 0 \le x \le 3 \\ 0 & \text{for all other } x \end{cases}$

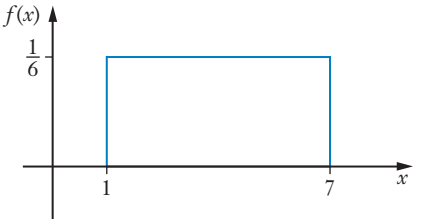
d $f(x) = \frac{x^2}{21}$ in the interval $1 \le x \le 4$

e $f(x) = \begin{cases} \frac{x}{8} & \text{for } 1 \le x \le 8 \\ 0 & \text{for all other } x \end{cases}$

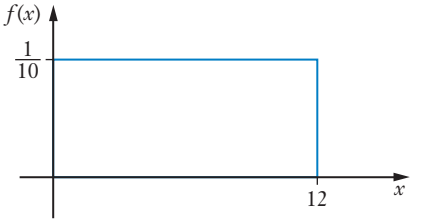
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3 Which of the following graphs are of probability density functions?

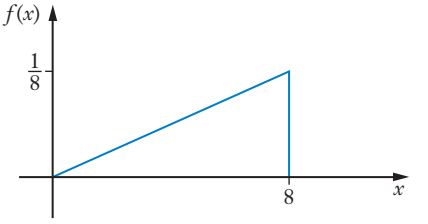
a



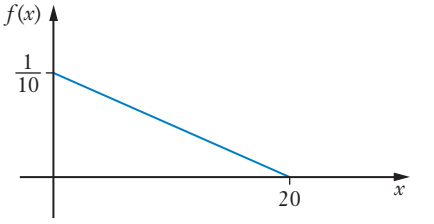
b



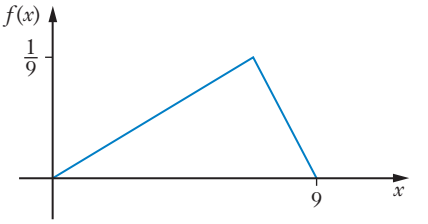
c



d



e



4 A probability density function has the equation $f(x) = \dfrac{x^4}{3355}$ over the domain $[2, b]$. Evaluate b.

5 Given the continuous probability distribution $f(x) = \begin{cases} kx^3 & \text{for } 0 \leq x \leq 5 \\ 0 & \text{for all other } x \end{cases}$

find the value of k.

6 A probability density function is given by $f(x) = ae^x$ over a certain domain. Find the exact value of a if the domain is:

a $[1, 3]$ **b** $[1, 7]$ **c** $[0, 4]$

7 A function is given by $f(x) = \dfrac{x^2}{72}$.

Over what domain starting at $x = 0$ is this a probability density function?

8 A PDF is given by $f(x) = \dfrac{2x^5}{87\,381}$ over the interval $1 \leq x \leq b$. Find the value of b.

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10.02 Calculating probabilities

Since $P(X = x) = 0$ for continuous probability distributions, we can only find the probability of a **range** of values $P(a \leq X \leq b)$.

Also, since $P(X = a) = 0$ and $P(X = b) = 0$, it makes no difference whether we use $\leq$ or $<$, $\geq$ or $>$.

$P(a < X < b) = P(a \leq X \leq b)$

WS Probability density functions

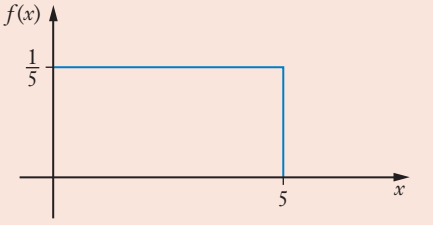
WS Uniform and triangular probability density functions

EXAMPLE 3

For the probability density function, find:

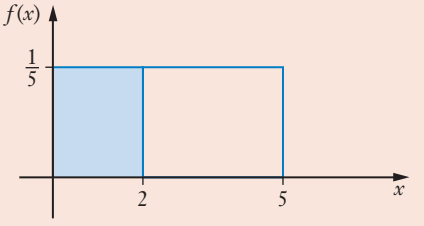
a $P(X \leq 2)$

b $P(1 < X < 4)$



Solution

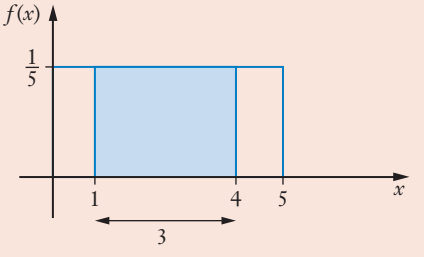
a $P(X \leq 2)$ is the shaded area between $x = 0$ and $x = 2$.



$$P(X \leq 2) = 2 \times \frac{1}{5}$$
$$= \frac{2}{5}$$

b $P(1 < X < 4)$ is the shaded area between $x = 1$ and $x = 4$.

Notice that $P(1 < X < 4) = P(1 \leq X \leq 4)$ since $P(X = 1) = P(X = 4) = 0$

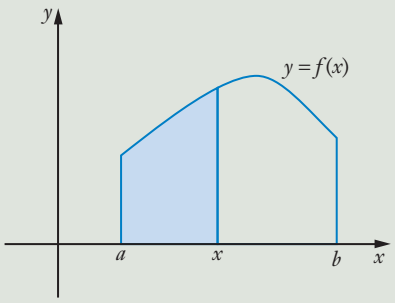


$$P(1 < X < 4) = 3 \times \frac{1}{5}$$
$$= \frac{3}{5}$$

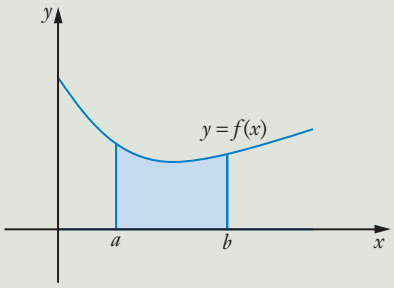
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Probabilities in probability density functions

$P(X \leq x) = \int_a^x f(x)\,dx$ where $y = f(x)$ is a PDF defined in the domain $[a, b]$.



$P(a \leq X \leq b) = \int_a^b f(x)\,dx$ where $y = f(x)$ is a PDF and a and b are in the defined domain.



EXAMPLE 4

A function is given by $f(x) = \dfrac{3x^2}{117}$ defined in the domain $[2, 5]$. Find:

a $P(X \leq 4)$ **b** $P(3 \leq X \leq 4)$

Solution

a $P(X \leq 4) = \int_2^4 \dfrac{3x^2}{117}\,dx$ (since domain is $[2, 5]$)

$$= \frac{1}{117}\int_2^4 3x^2\,dx$$

$$= \frac{1}{117}\left[x^3\right]_2^4$$

$$= \frac{1}{117}(4^3 - 2^3)$$

$$= \frac{56}{117}$$

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b $P(3 \le X \le 4) = \int_3^4 \frac{3x^2}{117}\,dx$

$$= \frac{1}{117}\int_3^4 3x^2\,dx$$

$$= \frac{1}{117}\left[x^3\right]_3^4$$

$$= \frac{1}{117}(4^3 - 3^3)$$

$$= \frac{37}{117}$$

Uniform distributions

In Year 11, Chapter 10, *Discrete probability distributions*, you learned that with a **uniform probability distribution**, every outcome has the same probability.

EXAMPLE 5

A continuous probability function $y = f(x)$ is uniform in the domain $[5, 15]$.

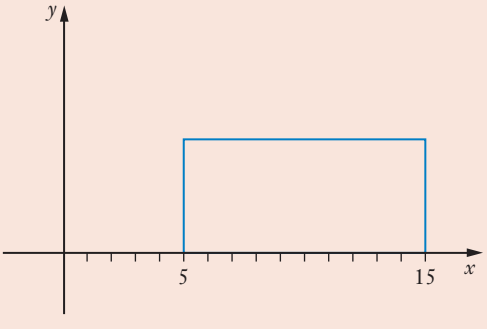
a Sketch the probability density function.

b Find:

i $P(X \ge 8)$ **ii** $P(7 \le X \le 10)$ **iii** $P(8 < X < 11)$

Solution

a A uniform distribution has all equal probabilities so will have the same height. This gives a rectangle.



PDF has area 1:

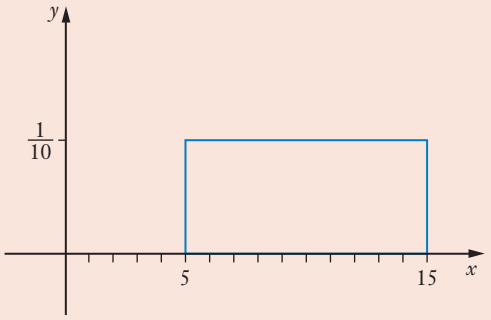
$A = bh$

$= (15 - 5) \times h$

$= 10h$

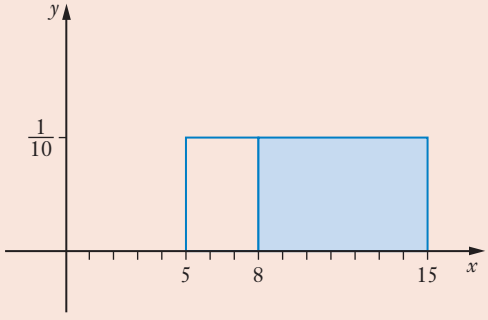
So $10h = 1$

$h = \frac{1}{10}$



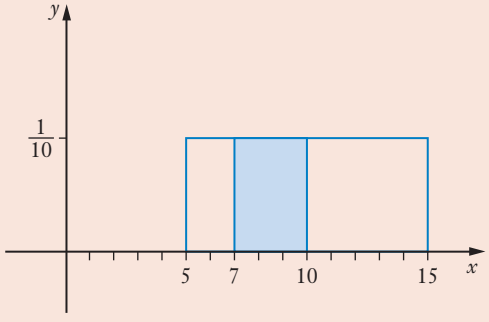
b **i** $P(X \geq 8) = (15 - 8) \times \frac{1}{10}$

$= \frac{7}{10}$



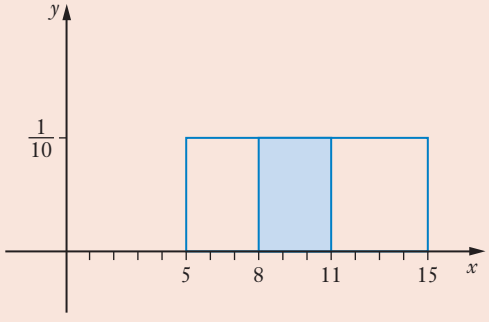
ii $P(7 \leq X \leq 10) = (10 - 7) \times \frac{1}{10}$

$= \frac{3}{10}$



iii $P(8 < X < 11) = (11 - 8) \times \frac{1}{10}$

$= \frac{3}{10}$

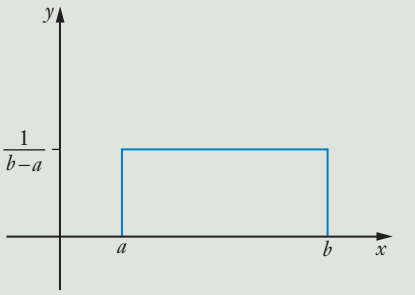


Notice in the example that intervals with the same width have the same probability.

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Uniform continuous probability distributions

$$f(x) = \begin{cases} \dfrac{1}{b-a} & \text{for } a \le x \le b \\ 0 & \text{for all other } x \text{ values} \end{cases}$$



Equal intervals along the x-axis will have the same probability.

Exercise 10.02 Calculating probabilities

1 For the continuous probability distribution graphed, find:

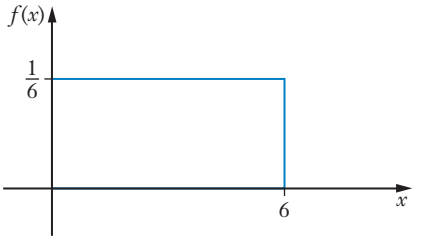
a $P(X \le 3)$

b $P(1 \le X \le 2)$

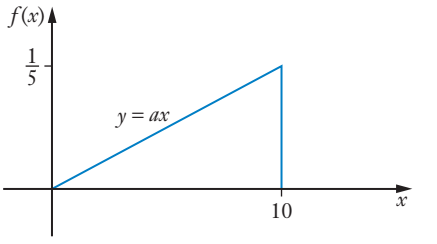
c $P(1 \le X \le 4)$

d $P(X < 4)$

e $P(X \ge 4)$



2 A probability density function is shown.



a Find the equation of the linear function $y = ax$.

b Find:

i $P(X < 9)$

ii $P(X \le 3)$

iii $P(4 \le X \le 7)$

iv $P(2 < X < 6)$

v $P(X > 5)$

3 The continuous probability distribution is defined by $f(x) = ax^2$ in the domain $[0, 5]$.

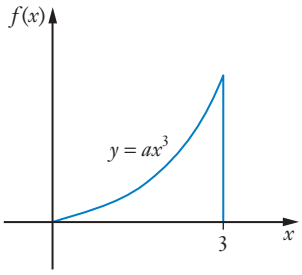
a Evaluate a.

b Find:

i $P(X \le 3)$ **ii** $P(1 < X < 4)$ **iii** $P(X > 2)$

iv $P(X < 1)$ **v** $P(3 \le X < 4)$

4 The continuous random variable X has the PDF shown.



a Evaluate a.

b Find:

i $P(1 \leq X \leq 3)$ **ii** $P(X < 2)$

iii $P(1 \leq X \leq 2)$ **iv** $P(X \leq 1)$

5 A continuous probability function is given by $f(x) = ke^x$, defined on the domain $[1, 6]$.

a Find the exact value of k.

b Find each exact probability:

i $P(2 \leq X \leq 5)$ **ii** $P(X < 4)$ **iii** $P(X \geq 3)$

6 a Show that $y = \sin x$ is a probability density function in the domain $\left[0, \frac{\pi}{2}\right]$.

b Find each exact probability:

i $P\left(X \leq \frac{\pi}{3}\right)$

ii $P\left(0 < X < \frac{\pi}{4}\right)$

iii $P\left(X > \frac{\pi}{6}\right)$

7 a Show that the uniform distribution $f(x) = \begin{cases} \frac{1}{b-a} & \text{for } a \leq x \leq b \\ 0 & \text{for all other } x \text{ values} \end{cases}$ is a probability density function.

b If $a = 3$ and $b = 7$, find:

i $P(X \leq 6)$ **ii** $P(X \geq 5)$ **iii** $P(5 \leq X \leq 6)$

WS Cumulative distribution function

10.03 Cumulative distribution function

In the previous section you integrated the PDF each time to find $P(X \leq x)$, a cumulative probability. The **cumulative distribution function** is a general formula for finding $P(X \leq x)$ directly.

Cumulative distribution function (CDF)

The cumulative distribution function is given by $F(x) = \int_a^x f(x)\,dx$ where $y = f(x)$ is a PDF defined in the domain $[a, b]$.

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EXAMPLE 6

A continuous probability function is given by $f(x) = \frac{4x^3}{255}$ defined in the domain $[1, 4]$.

a Find the cumulative distribution function.

b Use the CDF to find:

i $P(X \leq 3)$ **ii** $P(X < 1.6)$

Solution

a $F(x) = \int_a^x f(x)\,dx$ where $f(x) = \frac{4x^3}{255}$ is a PDF defined in the domain $[1, 4]$.

$$= \int_1^x \frac{4x^3}{255}\,dx$$

$$= \frac{1}{255}\int_1^x 4x^3\,dx$$

$$= \frac{1}{255}\left[x^4\right]_1^x$$

$$= \frac{1}{255}(x^4 - 1^4)$$

$$= \frac{1}{255}(x^4 - 1)$$

$$= \frac{x^4 - 1}{255}$$

b Using $F(x) = \frac{x^4 - 1}{255}$ to find $P(X \leq x)$:

i For $P(X \leq 3)$:

$$F(3) = \frac{3^4 - 1}{255}$$

$$= \frac{81 - 1}{255}$$

$$= \frac{80}{255}$$

$$= \frac{16}{51}$$

So $P(X \leq 3) = \frac{16}{51}$

ii For $P(X < 1.6)$:

$$F(1.6) = \frac{1.6^4 - 1}{255}$$

$$= 0.0218$$

So $P(X < 1.6) = 0.0218$

We can use the cumulative distribution function to find probabilities such as $P(X \geq a)$ or $P(a \leq X \leq b)$.

EXAMPLE 7

A continuous probability function is given by $f(x) = \dfrac{3x^2}{335}$ defined in the domain $[2, 7]$.

a Find the cumulative distribution function.

b Use the CDF to find:

i $P(X \geq 4)$ **ii** $P(3.5 \leq X \leq 6.2)$

Solution

a $F(x) = \int_a^x f(x)\,dx$ where $f(x) = \dfrac{3x^2}{335}$ is a PDF defined in the domain $[2, 7]$.

$$= \int_2^x \frac{3x^2}{335}\,dx$$

$$= \frac{1}{335}\int_2^x 3x^2\,dx$$

$$= \frac{1}{335}\left[x^3\right]_2^x$$

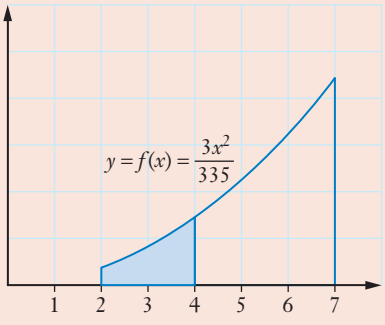
$$= \frac{1}{335}(x^3 - 2^3)$$

$$= \frac{1}{335}(x^3 - 8)$$

$$= \frac{x^3 - 8}{335}$$

b We use $F(x) = \dfrac{x^3 - 8}{335}$ to find $P(X \leq x)$.

i



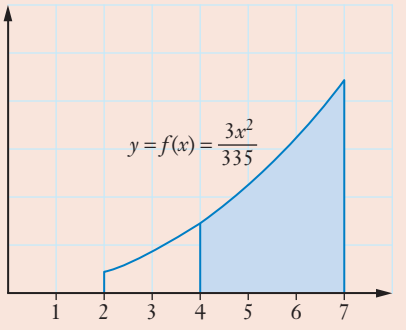
To find $P(X \geq 4)$, first find $P(X \leq 4)$:

The shaded part of the PDF is $P(X \leq 4)$.

$$P(X \leq 4) = \frac{4^3 - 8}{335}$$

$$= \frac{56}{335}$$

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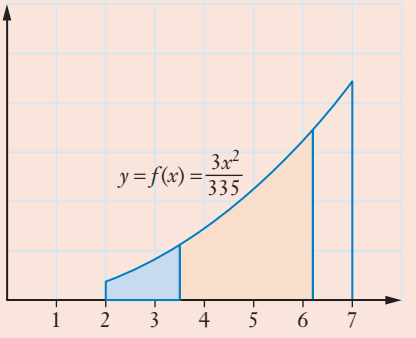
Area under the curve is 1:

$P(X \geq 4) = 1 - P(X \leq 4)$

$$= 1 - \frac{56}{335}$$

$$= \frac{279}{335}$$

ii



$P(X \leq 3.5)$ is the area to the left of $x = 3.5$

$P(X \leq 6.2)$ is the area to the left of $x = 6.2$

So $P(3.5 \leq X \leq 6.2) = P(X \leq 6.2) - P(X \leq 3.5)$

$$= \frac{6.2^3 - 8}{335} - \frac{3.5^3 - 8}{335}$$

$$= 0.688 - 0.104$$

$$= 0.583$$

Mode of a continuous probability distribution

We sometimes want to know what the highest probability is. This is the **mode**.

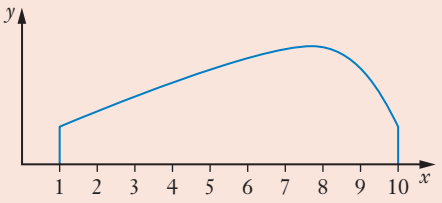
Mode

The mode is the maximum point of the probability density function.

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EXAMPLE 8

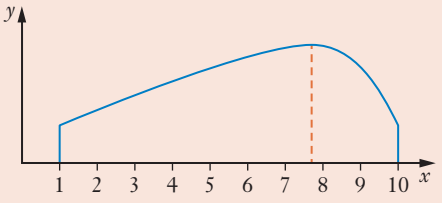
a Find the mode of the continuous probability distribution shown below.



b A continuous probability distribution is defined on the interval $1 \leq x \leq 5$ and has equation $f(x) = \dfrac{3x(6-x)}{92}$. Find the mode.

Solution

a



The highest point of the PDF is at $x = 7.7$

So the mode is 7.7.

b $f(x) = \dfrac{3x(6-x)}{92}$

$= \dfrac{3}{92}(6x - x^2)$

The function is a parabola with $a < 0$ so will have a maximum turning point.
We use calculus to see if this point lies within the defined domain [1, 5].
(We could also use $x = -\dfrac{b}{2a}$ for the axis of symmetry of a parabola).

$f'(x) = \dfrac{3}{92}(6 - 2x)$

For stationary points:

$f'(x) = 0$

$\dfrac{3}{92}(6 - 2x) = 0$

$6 - 2x = 0$

$6 = 2x$

$3 = x$

$x = 3$ lies in the domain [1, 5].

$f''(x) = \dfrac{3}{92}(-2)$

$= -\dfrac{3}{46}$

< 0

Concave down so a maximum turning point.

So the mode is 3.

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Exercise 10.03 Cumulative distribution function

1 Find the cumulative distribution function for each continuous probability distribution.

a $f(x) = \dfrac{x^2}{9}$ defined in the domain [0, 3]

b $f(x) = \dfrac{4x^3}{1296}$ defined in the domain [0, 6]

c $f(x) = \dfrac{e^x}{e^4 - 1}$ in the interval $0 \leq x \leq 4$

d $f(x) = \dfrac{4(x-2)^3}{625}$ in the domain [2, 7]

e $f(x) = \dfrac{3x(8-x)}{135}$ in the domain [2, 5]

2 **a** Find the cumulative distribution function for $f(x) = \begin{cases} \dfrac{5x^4}{7776} & \text{for } 1 \leq x \leq 6 \\ 0 & \text{for all other values} \end{cases}$.

b Find:

i $P(X \leq 3)$ **ii** $P(X \leq 2)$ **iii** $P(X < 5)$

iv $P(X > 4)$ **v** $P(2 \leq X \leq 4)$

3 A continuous probability distribution is given by $f(x) = \dfrac{4x^3}{2320}$ in the domain [3, 7].

a Find the cumulative distribution function.

b Find:

i $P(X \leq 4)$ **ii** $P(X \leq 6)$ **iii** $P(X \geq 5)$

iv $P(X > 4)$ **v** $P(4 \leq X < 6)$

4 A continuous probability distribution is defined by $f(x) = \dfrac{2e^{2x}}{e^{10} - 1}$ in the domain [0, 5].

a Find the cumulative distribution function.

b Calculate each probability correct to 2 significant figures.

i $P(X \leq 2)$ **ii** $P(X \leq 4)$ **iii** $P(X > 3)$

iv $P(X \geq 2.8)$ **v** $P(2 \leq X \leq 4)$

5 **a** Evaluate a if $f(x) = ax^3$ is a continuous probability distribution defined in the domain [0, 9].

b Find the cumulative distribution function.

c Find:

i $P(X \leq 5)$ **ii** $P(X \leq 4)$ **iii** $P(X > 8)$

iv $P(X \geq 3)$ **v** $P(2 \leq X \leq 6)$

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6 **a** Find the exact value of a if $f(x) = \frac{a}{x}$ is a continuous probability distribution defined in the domain $[1, 6]$.

b Find the cumulative distribution function.

c Find to 2 decimal places:

i $P(X \leq 3)$ **ii** $P(X \leq 2)$ **iii** $P(X > 5)$

iv $P(X \geq 4)$ **v** $P(2 \leq X \leq 5)$

7 **a** Show that $y = \cos x$ is a probability density function in the domain $\left[\frac{3\pi}{2}, 2\pi\right]$.

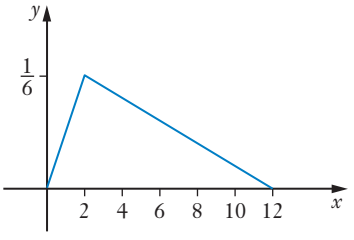
b Find the cumulative distribution function.

c Find each probability in exact form:

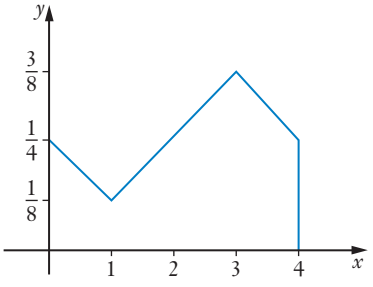
i $P\left(X \leq \frac{5\pi}{3}\right)$ **ii** $P\left(X \geq \frac{7\pi}{4}\right)$ **iii** $P\left(\frac{5\pi}{3} \leq X \leq \frac{11\pi}{6}\right)$

8 Find the mode of each continuous probability distribution.

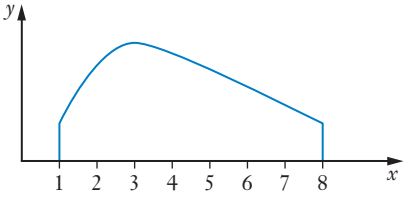
a



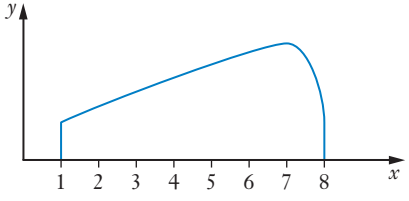
b



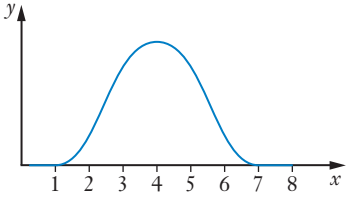
c



d



e



f $f(x) = -\frac{3(x^2 - 8x - 9)}{434}$ defined in the domain $[0, 7]$

g $f(x) = \frac{4e^{4x}}{e^8(e^{16} - 1)}$ defined in the interval $2 \leq x \leq 6$

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h $f(x) = -\dfrac{3(x^2 - 16x + 15)}{1100}$ defined in the domain $[1, 11]$

i $f(x) = \dfrac{2(2x^3 - 33x^2 + 168x + 3)}{2105}$ defined in the interval $0 \leq x \leq 5$

j $f(x) = \dfrac{3x^2}{342}$ defined in the interval $1 \leq x \leq 7$

9 **a** Find the mode of the function $f(x) = -\dfrac{3}{22}(x^2 - 6x + 5)$ defined on the domain $[2, 4]$.

b Find the cumulative distribution function.

c Find $P(X \leq a)$ where a is the mode.

10 The times that athletes took to finish a race varied between 3 and 7 minutes and are represented by the continuous probability function $f(x) = \dfrac{1}{116}(x^3 - 9x^2 + 24x + 1)$ defined in the domain $[3, 7]$.

a Find the cumulative distribution function.

b Find the probability that an athlete will finish this race:

i in less than 5 minutes

ii in 4 minutes or more

iii in between 4 and 5 minutes

c What is the most likely time in which an athlete would finish the race?

10.04 Quantiles

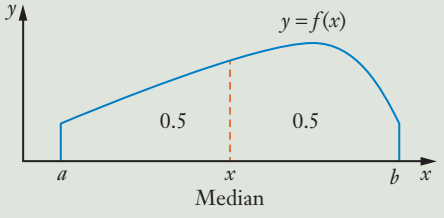
WS Quantiles

Median

For a continuous probability distribution, the **median** is the value of x that splits the distribution into halves. Because the PDF has an area of 1, the area on each side of the median is $\frac{1}{2}$.

Median

The median lies at the point x where $\int_a^x f(x)\,dx = 0.5$ given $y = f(x)$ is a PDF defined in the domain $[a, b]$.



Median

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The median of a continuous probability distribution

EXAMPLE 9

Find the median of the continuous probability distribution defined as $f(x) = \frac{x^2}{21}$ in the domain [1, 4].

Solution

For $f(x) = \frac{x^2}{21}$ defined in the domain [1, 4], first find the cumulative distribution function (CDF):

$$\int_1^x \frac{x^2}{21}\,dx = \frac{1}{21}\int_1^x x^2\,dx$$

$$= \frac{1}{21}\left[\frac{x^3}{3}\right]_1^x$$

$$= \frac{1}{21}\left(\frac{x^3}{3} - \frac{1^3}{3}\right)$$

$$= \frac{1}{21}\left(\frac{x^3 - 1}{3}\right)$$

$$= \frac{x^3 - 1}{63}$$

For the median:

$$\int_a^x f(x)\,dx = 0.5$$

$$\frac{x^3 - 1}{63} = 0.5$$

$$x^3 - 1 = 31.5$$

$$x^3 = 32.5$$

$$x = \sqrt[3]{32.5}$$

$$\approx 3.2$$

So the median is 3.2.

This means $P(X < 3.2) = 0.5$.

Quartiles, deciles and percentiles

You learned about quartiles, deciles and percentiles in Chapter 7, *Statistics*. They are values that separate a proportion of a set of data. For example, Q_1 > bottom 25% of scores, Q_3 > bottom 75% of scores, 2nd decile > bottom 20% of scores and 67th percentile > bottom 67% of scores.

EXAMPLE 10

A continuous probability distribution is defined as $f(x) = \frac{x^4}{11\,605}$ in the domain [4, 9].

Find, correct to 2 decimal places:

a the 1st quartile

b the 38th percentile

c the 7th decile

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Solution

First, find the CDF.

$$\int_4^x \frac{x^4}{11\,605}\,dx = \frac{1}{11605}\int_4^x x^4\,dx$$

$$= \frac{1}{11\,605}\left[\frac{x^5}{5}\right]_4^x$$

$$= \frac{1}{11\,605}\left(\frac{x^5}{5} - \frac{4^5}{5}\right)$$

$$= \frac{1}{11\,605}\left(\frac{x^5 - 1024}{5}\right)$$

$$= \frac{x^5 - 1024}{58\,025}$$

a 1st quartile: 25%

$$\int_a^x f(x)\,dx = 0.25$$

$$\frac{a^5 - 1024}{58\,025} = 0.25$$

$$a^5 - 1024 = 14\,506.25$$

$$a^5 = 15\,530.25$$

$$a \approx 6.89$$

So the 1st quartile is 6.89.

This means $P(X < 6.89) = 0.25$.

b 38th percentile: 38%

$$\int_a^x f(x)\,dx = 0.38$$

$$\frac{a^5 - 1024}{58\,025} = 0.38$$

$$a^5 - 1024 = 22\,049.5$$

$$a^5 = 23\,073.5$$

$$a \approx 7.46$$

So the 38th percentile is 7.46.

This means $P(X < 7.46) = 0.38$.

c 7th decile: 70%

$$\int_a^x f(x)\,dx = 0.7$$

$$\frac{a^5 - 1024}{58\,025} = 0.7$$

$$a^5 - 1024 = 40\,617.5$$

$$a^5 = 41\,641.5$$

$$a \approx 8.39$$

So the 7th decile is 8.39.

This means $P(X < 8.39) = 0.7$.

Exercise 10.04 Quantiles

1 Find the median of each continuous random variable correct to 2 decimal places.

a $f(x) = \dfrac{3x^2}{511}$ defined on the interval $1 \leq x \leq 8$

b $f(x) = \dfrac{4x^3}{2401}$ defined in the domain $[0, 7]$

c $f(x) = \dfrac{5x^4}{16\,807}$ in the interval $0 \leq x \leq 7$

d $f(x) = \dfrac{3(x-3)^2}{16}$ in the domain $[1, 5]$

e $f(x) = \dfrac{(3x+1)^2}{244}$ in the interval $0 \leq x \leq 4$

f $f(x) = \dfrac{4x^3}{6560}$ defined in the domain $[1, 9]$

g $f(x) = \dfrac{3x^2}{1034}$ defined in the domain $[3, 11]$

h $f(x) = \dfrac{6x^5}{15\,625}$ in the interval $0 \leq x \leq 5$

i $f(x) = \dfrac{(2x-1)^4}{16\,105}$ in the domain $[1, 6]$

j $f(x) = \dfrac{x(x^2-3)^3}{3570}$ defined in the domain $[2, 4]$

2 For each continuous probability distribution, find:

i the 1st quartile **ii** the 2nd decile **iii** the 77th percentile

a $f(x) = \dfrac{3x^2}{973}$ defined in the domain $[3, 10]$

b $f(x) = \dfrac{x^3}{324}$ defined in the interval $0 \leq x \leq 6$

c $f(x) = \dfrac{5x^4}{3124}$ defined in the interval $1 \leq x \leq 5$

3 For the continuous probability distribution $f(x) = \dfrac{3x^2}{512}$ defined in the domain $[0, 8]$, find:

a the median

b the 35th percentile

4 For the continuous probability distribution $f(x) = \dfrac{x^2}{168}$ defined on the interval $2 \leq x \leq 8$, find:

a the median **b** the 1st quartile **c** the 3rd quartile

d the 67th percentile **e** the 14th percentile **f** the 8th decile

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5 For the continuous probability distribution defined as $f(x) = \frac{x^2}{576}$ on the interval $0 \leq x \leq 12$, find:

a the 20th percentile

b the median

c the 3rd quartile

6 For the continuous probability distribution $f(x) = \frac{x^3}{1020}$ defined in the interval $2 \leq x \leq 8$, find:

a the cumulative probability function

b $P(X \leq 5)$

c $P(X > 4)$

d $P(3 \leq X \leq 7)$

e the median

f the 3rd quartile

g the 9th decile

h the 23rd percentile

10.05 Normal distribution

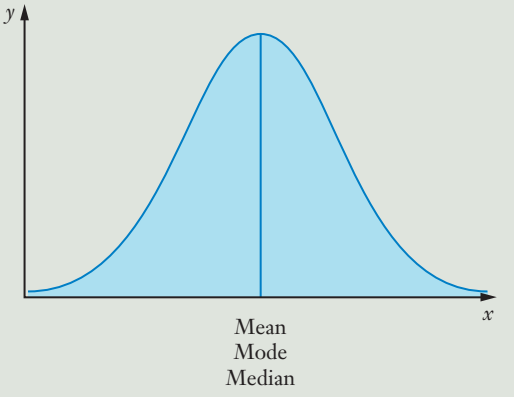
WS
The normal distribution

The **normal distribution** is a special continuous probability distribution. Its probability density function is often called a **bell curve** because of its shape.

Normal distribution

The normal distribution is a symmetrical bell-shaped function.

The mean, mode and median are equal, at the centre of the probability density function.



There are many examples of data that are normally distributed, such as IQ, birth weights, ages, reaction times and exam results in a school.

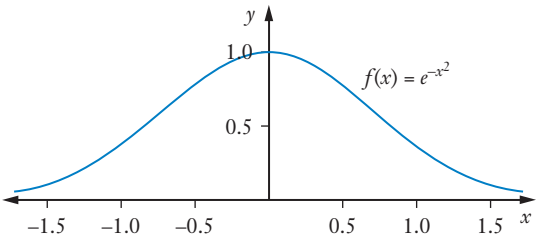
We use the population mean μ and standard deviation σ for the normal distribution.

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TECHNOLOGY

The normal distribution

A good estimate for the **shape** of the normal distribution is $f(x) = e^{-x^2}$.



Use your graphing techniques and technology to sketch this function.

Given a population that is normally distributed with mean $\mu = 10$ and standard deviation $\sigma = 2$, use technology to sketch the graph of

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

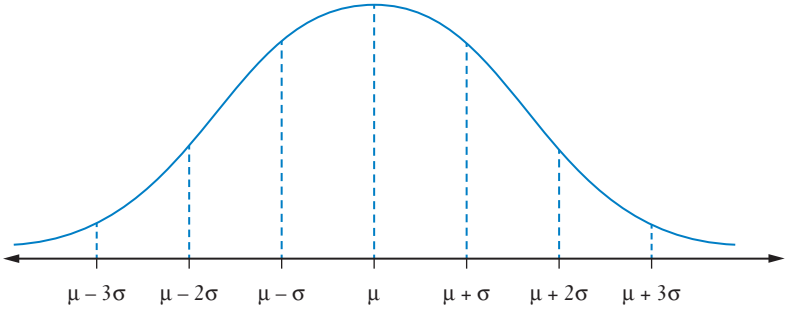
This is the actual equation of the normal distribution.

What do you notice about this graph? Use technology and the trapezoidal rule with many subintervals to approximate the area under the graph. What do you find?

Research the normal distribution and its features.

Graphing a normal distribution

In a normal distribution, most of the data lies within 3 standard deviations of the mean, with the mean in the centre.

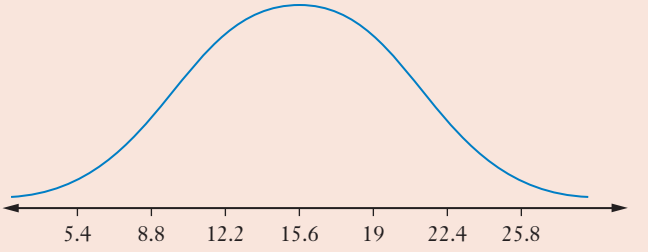


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EXAMPLE 11

a A set of data is normally distributed with mean 8.3 and standard deviation 1.2. Sketch the probability distribution function.

b A normal distribution has the probability density function below. Find its mean and standard deviation.



Solution

a

$\mu + \sigma = 8.3 + 1.2$

$= 9.5$

$\mu + 2\sigma = 8.3 + 2(1.2)$

$= 10.7$

$\mu + 3\sigma = 8.3 + 3(1.2)$

$= 11.9$

$\mu - \sigma = 8.3 - 1.2$

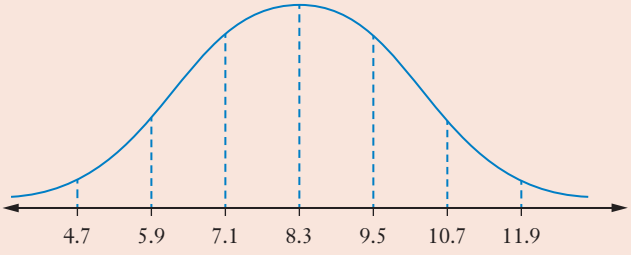
$= 7.1$

$\mu - 2\sigma = 8.3 - 2(1.2)$

$= 5.9$

$\mu - 3\sigma = 8.3 - 3(1.2)$

$= 4.7$



b

$\mu = 15.6$ (centre of the distribution)

$\mu + \sigma = 19$

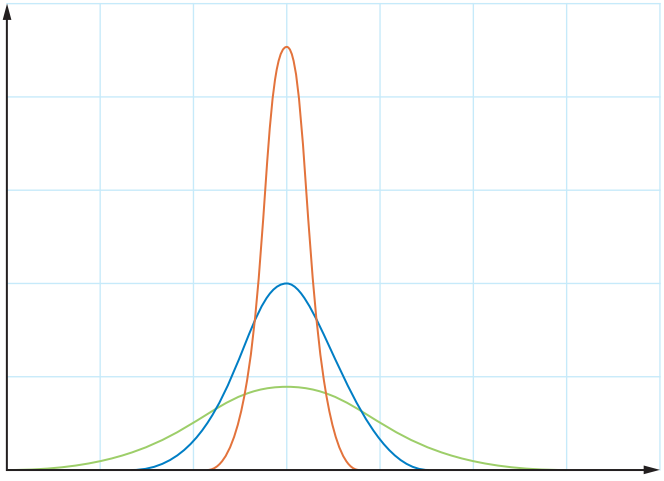
$15.6 + \sigma = 19$

$\sigma = 19 - 15.6$

$= 3.4$ (check that all the differences between all values on the graph are 3.4)

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The normal distribution can have different shapes depending on the size of the standard deviation. In the diagram, the green curve shows the normal distribution with the highest standard deviation.



Standard normal distribution

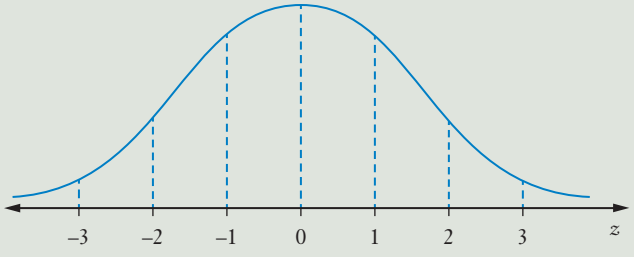
It is difficult to find probabilities in the normal distribution by integration because the equation of the cumulative distribution function is complicated. To get around this, tables of probabilities have been developed from the **standard normal distribution**. This is a normal curve that has been transformed so that the mean is 0 and the standard deviation is 1. The values of a standard normal distribution are called z rather than x, also known as **z-scores** or **standardised scores**.

Standard normal distribution

$\mu = 0$

$\sigma = 1$

Area under the curve is 1.



Normal distribution table

Probability tables for the standard normal distribution

Probability tables for the standard normal distribution begin next page. A copy can also be downloaded from *NelsonNet*. Values represent the area to the left of (or less than) the z-score. Row labels show the z-score to one decimal place. Column labels show the second decimal place.

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z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
–3.4	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0002
–3.3	0.0005	0.0005	0.0005	0.0004	0.0004	0.0004	0.0004	0.0004	0.0004	0.0003
–3.2	0.0007	0.0007	0.0006	0.0006	0.0006	0.0006	0.0006	0.0005	0.0005	0.0005
–3.1	0.0010	0.0009	0.0009	0.0009	0.0008	0.0008	0.0008	0.0008	0.0007	0.0007
–3.0	0.0013	0.0013	0.0013	0.0012	0.0012	0.0011	0.0011	0.0011	0.0010	0.0010
–2.9	0.0019	0.0018	0.0018	0.0017	0.0016	0.0016	0.0015	0.0015	0.0014	0.0014
–2.8	0.0026	0.0025	0.0024	0.0023	0.0023	0.0022	0.0021	0.0021	0.0020	0.0019
–2.7	0.0035	0.0034	0.0033	0.0032	0.0031	0.0030	0.0029	0.0028	0.0027	0.0026
–2.6	0.0047	0.0045	0.0044	0.0043	0.0041	0.0040	0.0039	0.0038	0.0037	0.0036
–2.5	0.0062	0.0060	0.0059	0.0057	0.0055	0.0054	0.0052	0.0051	0.0049	0.0048
–2.4	0.0082	0.0080	0.0078	0.0075	0.0073	0.0071	0.0069	0.0068	0.0066	0.0064
–2.3	0.0107	0.0104	0.0102	0.0099	0.0096	0.0094	0.0091	0.0089	0.0087	0.0084
–2.2	0.0139	0.0136	0.0132	0.0129	0.0125	0.0122	0.0119	0.0116	0.0113	0.0110
–2.1	0.0179	0.0174	0.0170	0.0166	0.0162	0.0158	0.0154	0.0150	0.0146	0.0143
–2.0	0.0228	0.0222	0.0217	0.0212	0.0207	0.0202	0.0197	0.0192	0.0188	0.0183
–1.9	0.0287	0.0281	0.0274	0.0268	0.0262	0.0256	0.0250	0.0244	0.0239	0.0233
–1.8	0.0359	0.0351	0.0344	0.0336	0.0329	0.0322	0.0314	0.0307	0.0301	0.0294
–1.7	0.0446	0.0436	0.0427	0.0418	0.0409	0.0401	0.0392	0.0384	0.0375	0.0367
–1.6	0.0548	0.0537	0.0526	0.0516	0.0505	0.0495	0.0485	0.0475	0.0465	0.0455
–1.5	0.0668	0.0655	0.0643	0.0630	0.0618	0.0606	0.0594	0.0582	0.0571	0.0559
–1.4	0.0808	0.0793	0.0778	0.0764	0.0749	0.0735	0.0721	0.0708	0.0694	0.0681
–1.3	0.0968	0.0951	0.0934	0.0918	0.0901	0.0885	0.0869	0.0853	0.0838	0.0823
–1.2	0.1151	0.1131	0.1112	0.1093	0.1075	0.1056	0.1038	0.1020	0.1003	0.0985
–1.1	0.1357	0.1335	0.1314	0.1292	0.1271	0.1251	0.1230	0.1210	0.1190	0.1170
–1.0	0.1587	0.1562	0.1539	0.1515	0.1492	0.1469	0.1446	0.1423	0.1401	0.1379
–0.9	0.1841	0.1814	0.1788	0.1762	0.1736	0.1711	0.1685	0.1660	0.1635	0.1611
–0.8	0.2119	0.2090	0.2061	0.2033	0.2005	0.1977	0.1949	0.1922	0.1894	0.1867
–0.7	0.2420	0.2389	0.2358	0.2327	0.2296	0.2266	0.2236	0.2206	0.2177	0.2148
–0.6	0.2743	0.2709	0.2676	0.2643	0.2611	0.2578	0.2546	0.2514	0.2483	0.2451
–0.5	0.3085	0.3050	0.3015	0.2981	0.2946	0.2912	0.2877	0.2843	0.2810	0.2776
–0.4	0.3446	0.3409	0.3372	0.3336	0.3300	0.3264	0.3228	0.3192	0.3156	0.3121
–0.3	0.3821	0.3783	0.3745	0.3707	0.3669	0.3632	0.3594	0.3557	0.3520	0.3483
–0.2	0.4207	0.4168	0.4129	0.4090	0.4052	0.4013	0.3974	0.3936	0.3897	0.3859
–0.1	0.4602	0.4562	0.4522	0.4483	0.4443	0.4404	0.4364	0.4325	0.4286	0.4247
–0.0	0.5000	0.4960	0.4920	0.4880	0.4840	0.4801	0.4761	0.4721	0.4681	0.4641

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z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990
3.1	0.9990	0.9991	0.9991	0.9991	0.9992	0.9992	0.9992	0.9992	0.9993	0.9993
3.2	0.9993	0.9993	0.9994	0.9994	0.9994	0.9994	0.9994	0.9995	0.9995	0.9995
3.3	0.9995	0.9995	0.9995	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	0.9997
3.4	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9998

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EXAMPLE 12

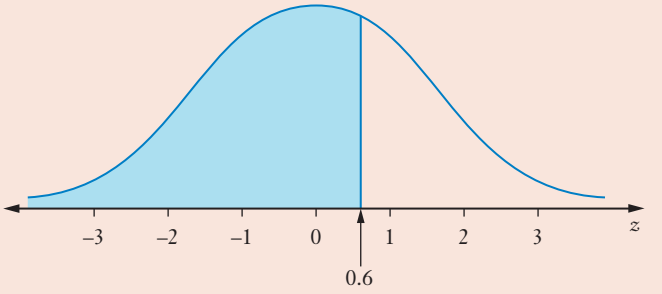
For a standard normal distribution, use the table to find:

a $P(z \leq 0.6)$ **b** $P(z \leq -1.83)$ **c** $P(z < 2.34)$

d $P(z \geq -2.7)$ **e** $P(-0.3 \leq z \leq 1.4)$

Solution

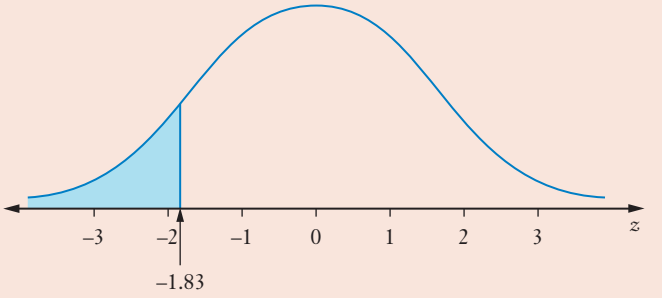
a The table gives the area under the PDF for the standard normal distribution.



Find 0.6 in the left column of the table.

$P(z \leq 0.6) = 0.7257$

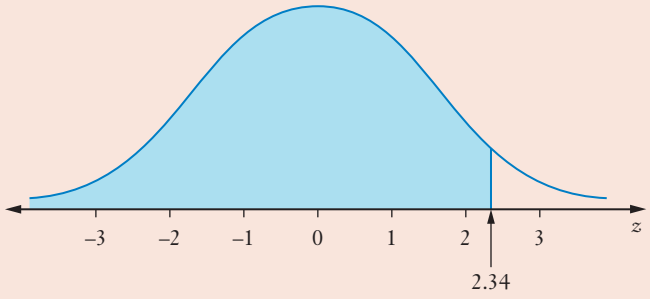
b



For –1.83, find –1.8 in the left column of the table and the entry under 0.03 in this row.

$P(z \leq -1.83) = 0.0336$

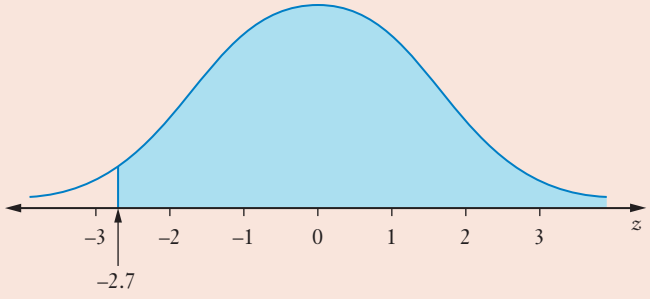
c



For 2.34, find 2.3 in the left column of the table and the entry under 0.04 in this row.

$P(z < 2.34) = 0.9904$

d



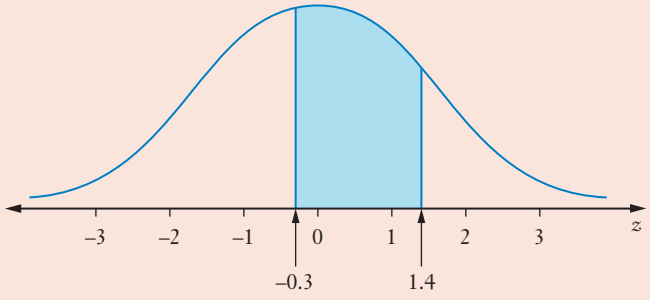
Find –2.7 in the left column of the table.

$P(z \leq -2.7) = 0.0035$

$P(z \geq -2.7) = 1 - 0.0035$

$= 0.9965$

e



$P(-0.3 \leq z \leq 1.4) = P(z \leq 1.4) - P(z \leq -0.3)$

$= 0.9192 - 0.3821$

$= 0.5371$

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Quartiles, deciles and percentiles

EXAMPLE 13

For a standard normal distribution, use the table on pages 441–442 to find:

a the median **b** the lower quartile **c** the upper quartile

d the 84th percentile **e** the 6th decile

Solution

a The median separates the bottom half of the scores.

We want to find the score with the cumulative probability of 0.5 in the table.

Median = 0 (this is the same value as the mean)

b The lower quartile separates the bottom 0.25 of the data.

In the table there are 2 values close to 0.25:

$P(z \leq -0.67) = 0.2514$ and $P(z \leq -0.68) = 0.2483$

0.2514 is closer to 0.25 so the lower quartile is approximately –0.67.

(Note: 0.2514 – 0.25 = 0.0014, and 0.25 – 0.2483 = 0.0017.)

c The upper quartile separates the bottom 0.75 of the data.

In the table there are 2 values close to 0.75:

$P(z \leq 0.67) = 0.7486$ and $P(z \leq 0.68) = 0.7517$

0.7486 is closer to 0.75 so the upper quartile is approximately 0.67.

Notice that the values for the lower and upper quartiles are ±0.67 because the normal distribution is symmetrical.

d The 84th percentile separates the bottom 0.84 of the data.

In the table there are 2 values close to 0.84:

$P(z \leq 0.99) = 0.8389$ and $P(z \leq 1) = 0.8413$

0.8389 is closer to 0.84 so the 84th percentile is approximately 0.99.

e The 6th decile separates the bottom 0.6 of the data.

In the table there are 2 values close to 0.6:

$P(z \leq 0.25) = 0.5987$ and $P(z \leq 0.26) = 0.6026$

0.5897 is closer to 0.6 so the 6th decile is approximately 0.25.

EXAMPLE 14

Shade the area of the normal distribution where the values are:

a above the top 20% of data **b** above the top 10% of data

Solution

a $P(z > a) = 20\%$, so $P(z \leq a) = 80\%$ (0.8 or the 8th decile)

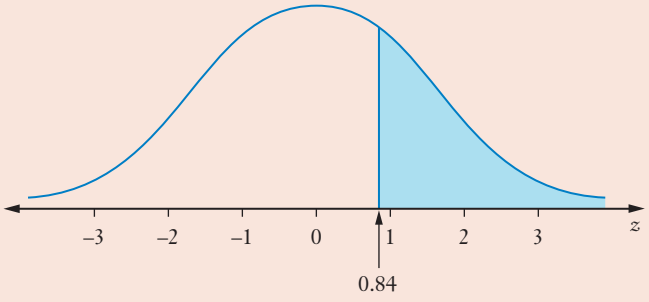
From the table:

$P(z \leq 0.84) = 0.7995$ and $P(z \leq 0.85) = 0.8023$

0.7995 is closer to 0.8 so the 8th decile is approximately 0.84

This means all values to the left of 0.84 lie below 80%.

So the top 20% lies to the right of 0.84.



b $P(z > a) = 10\%$, so $P(z \leq a) = 90\%$ (0.9 or the 9th decile)

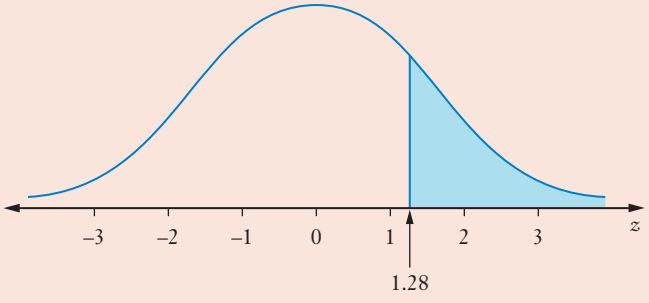
From the table:

$P(z \leq 1.28) = 0.8997$ and $P(z \leq 1.29) = 0.9015$

0.8997 is closer to 0.9 so the 9th decile is approximately 1.28.

This means all values to the left of 1.28 lie below 90%.

So the top 10% lies to the right of 1.28.



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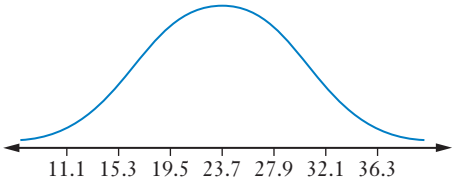
Exercise 10.05 Normal distribution

1 Draw a probability density function for the normal distribution with:

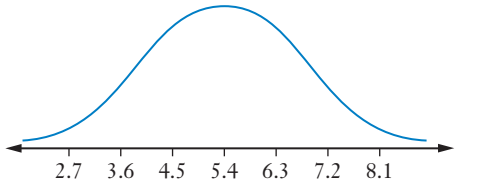
- **a** mean 9 and standard deviation 2
- **b** mean 8.6 and standard deviation 0.3
- **c** mean 11.5 and standard deviation 1.4
- **d** mean 27 and standard deviation 2.5
- **e** mean 115.2 and standard deviation 3.2

2 What is the mean and standard deviation of each normal distribution?

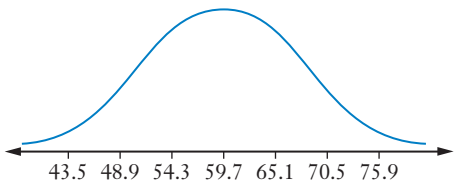
a



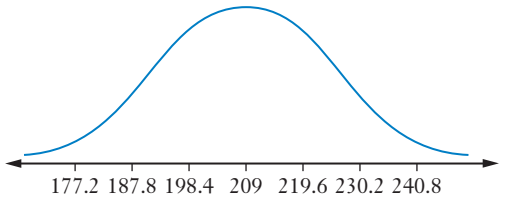
b



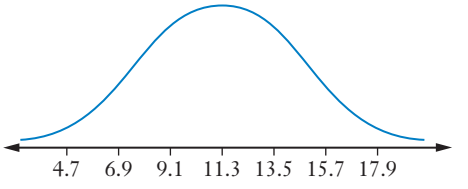
c



d



e



3 Draw the probability density function for a standard normal distribution.

4 Use the probability table for a standard normal distribution on pages 441–442 to find:

- **a** $P(z \leq 0)$
- **b** $P(z \leq 1)$
- **c** $P(z \leq 2)$
- **d** $P(z \leq 3)$
- **e** $P(z \leq -1)$
- **f** $P(z \leq -2)$
- **g** $P(z \leq -3)$
- **h** $P(z \leq 1.5)$
- **i** $P(z < -2.67)$
- **j** $P(z \leq 3.09)$

5 Use the probability table to find:

- **a** $P(z \geq -0.46)$
- **b** $P(z > 2.11)$
- **c** $P(z \geq -2.01)$
- **d** $P(-2.4 \leq z \leq -1.76)$
- **e** $P(-2.2 \leq z \leq 2.2)$
- **f** $P(1.21 < z < 1.89)$
- **g** $P(-1.45 \leq z \leq 3.1)$
- **h** $P(-1 \leq z \leq 1)$
- **i** $P(-2 \leq z \leq 2)$
- **j** $P(-3 \leq z \leq 3)$

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6 Use the probability table to find:

a the 8th decile
b the 3rd quartile
c the 29th percentile
d the 2nd decile
e the 89th percentile
f the 12th percentile
g the 3rd decile
h the 1st quartile
i the 63rd percentile

7 Sketch a normal curve and draw on it the area where values lie in the:

a bottom 30%
b bottom 15%
c top 67%
d top 24%
e top 12%



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10.06 Empirical rule

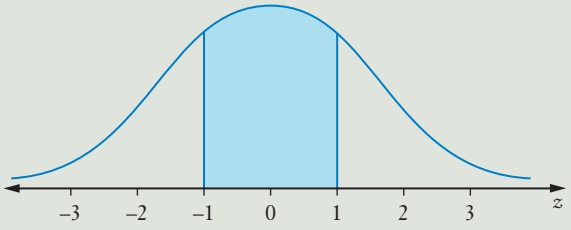
WS The empirical rule

In Question 5 of the previous exercise you calculated $P(-1 \leq z \leq 1)$, $P(-2 \leq z \leq 2)$ and $P(-3 \leq z \leq 3)$, the probability that values in a normal distribution will fall within 1, 2 or 3 standard deviations of the mean respectively. These probabilities are part of the **empirical rule**.

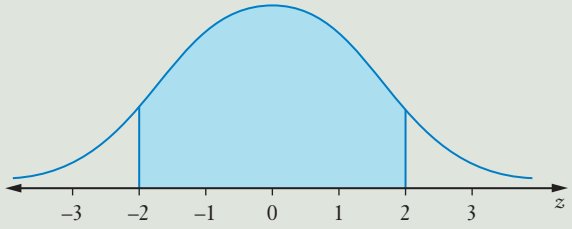
WS The normal curve

Empirical rule

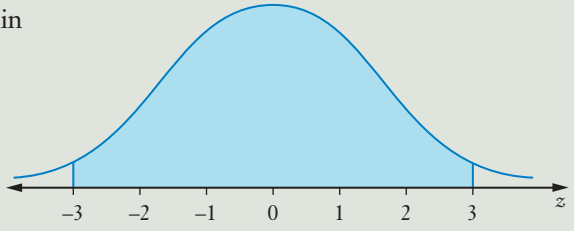
Approximately 68% of scores lie within 1 standard deviation of the mean, with z-scores between -1 and 1.



Approximately 95% of scores lie within 2 standard deviations of the mean, with z-scores between -2 and 2.



Approximately 99.7% of scores lie within 3 standard deviations of the mean, with z-scores between -3 and 3.



EXAMPLE 15

a A data set is normally distributed with mean 20 and standard deviation 3.

What percentage of scores lie between:

i 17 and 23? **ii** 14 and 26? **iii** 11 and 29?

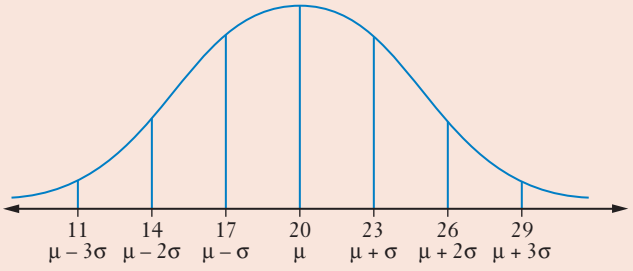
b A normal distribution has $\mu = 65.2$ and $\sigma = 1.3$. What percentage of scores lie between:

i 61.3 and 65.2? **ii** 65.2 and 66.5? **iii** 62.6 and 69.1?

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Solution

a We can draw the PDF for the normal curve.



i Scores between 17 and 23 are within 1 standard deviation of the mean.

So about 68% of scores lie between 17 and 23.

ii Scores between 14 and 26 are within 2 standard deviations of the mean.

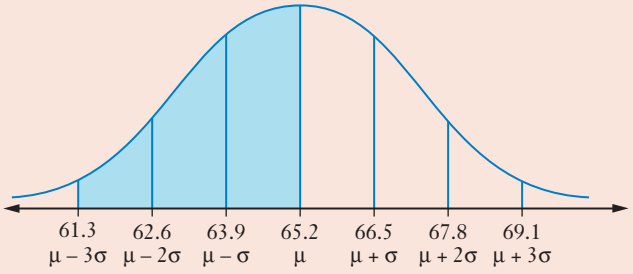
So about 95% of scores lie between 14 and 26.

iii Scores between 11 and 29 are within 3 standard deviations of the mean.

So about 99.7% of scores lie between 11 and 29.

b We can draw the PDF for the normal curve.

i



Scores between 61.3 and 69.1 are within 3 standard deviations of the mean (99.7% of data).

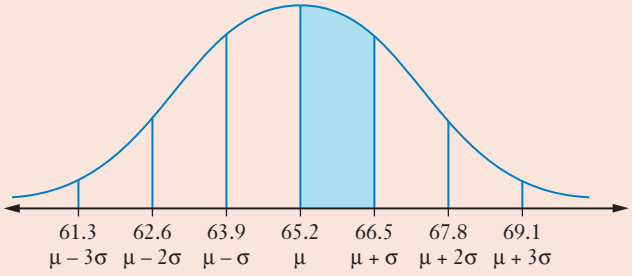
Scores between 61.3 and 65.2 are half this area.

$99.7\% \div 2 = 49.85\%$

So 49.85% of scores lie between 61.3 and 65.2.

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ii



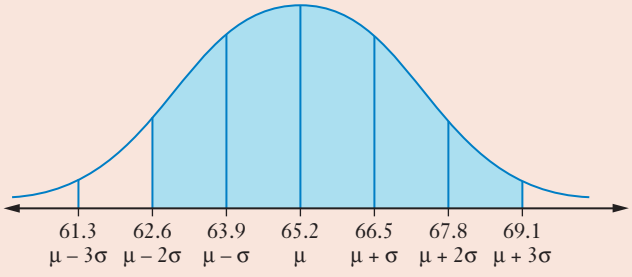
Scores between 63.9 and 66.5 are within 1 standard deviation of the mean (68% of data).

Scores between 65.2 and 66.5 are half this area.

$68\% \div 2 = 34\%$

So 34% of scores lie between 65.2 and 66.5.

iii



Scores between 62.6 and 65.2 are half the area within 2 standard deviations of the mean $\left(\frac{1}{2} \times 95\%\right)$.

Scores between 65.2 and 69.1 are half the area within 3 standard deviations of the mean $\left(\frac{1}{2} \times 99.7\%\right)$.

$$\text{Total area} = \frac{1}{2} \times 95\% + \frac{1}{2} \times 99.7\%$$
$$= 47.5\% + 49.85\%$$
$$= 97.35\%$$

So 97.35% of scores lie between 62.6 and 69.1.

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Exercise 10.06 Empirical rule

1 What is the approximate percentage of data in a normal distribution that lies within:

a 1 standard deviation of the mean?

b 2 standard deviations of the mean?

c 3 standard deviations of the mean?

2 A set of data has a mean of 15 and a standard deviation of 1.5. If the data set is normally distributed, find the percentage of data that lies between:

a 13.5 and 16.5 **b** 12 and 18 **c** 10.5 and 19.5

3 A set of data is normally distributed with a mean of 8.4 and a standard deviation of 0.9. Find the percentage of data that lies between:

a 7.5 and 9.3 **b** 6.6 and 10.2 **c** 5.7 and 11.1

4 A set of data is normally distributed with mean 18 and standard deviation 2. Find the percentage of data that lies between:

a 16 and 20 **b** 14 and 22 **c** 12 and 24

d 16 and 18 **e** 18 and 24 **f** 12 and 22

5 A normal distribution has a mean of 65 and standard deviation 4.

a Sketch the graph of its PDF.

b Find the percentage of data that lies between:

i 57 and 73 **ii** 61 and 65 **iii** 65 and 77

iv 57 and 69 **v** 61 and 73

6 A normal distribution has a mean of 9.7 and standard deviation 2.1. Find the percentage of data that lies between:

a 7.6 and 11.8 **b** 9.7 and 11.8 **c** 9.7 and 13.9

d 5.5 and 9.7 **e** 3.4 and 11.8

WS The standard normal curve

WS *z*-scores

WS Areas under the normal curve

z-scores

10.07 *z*-scores

We can transform any normal distribution into a standard normal distribution by using *z*-scores.

z-scores

To convert a raw score, x, into a z-score, use the formula:

$$z = \frac{x - \mu}{\sigma}$$

where μ is the mean and σ is the standard deviation of the distribution.

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EXAMPLE 16

A data set is normally distributed with mean 15 and standard deviation 2.

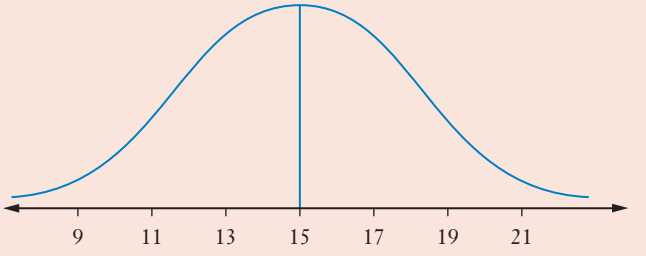
a Draw the probability density function for this distribution.

b Use z-scores to convert each x value to standardised scores:

i $x = 19$ **ii** $x = 15$ **iii** $x = 9$

Solution

a Drawing the PDF gives the curve below.



b $\mu = 15$ and $\sigma = 2$

i For $x = 19$:

$$z = \frac{x - \mu}{\sigma}$$
$$= \frac{19 - 15}{2}$$
$$= 2$$

This is because 19 is 2 standard deviations above the mean.

ii For $x = 15$:

$$z = \frac{x - \mu}{\sigma}$$
$$= \frac{15 - 15}{2}$$
$$= 0$$

This is because 15 is the mean.

iii For $x = 9$:

$$z = \frac{x - \mu}{\sigma}$$
$$= \frac{9 - 15}{2}$$
$$= -3$$

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EXAMPLE 17

A data set is normally distributed with mean 23.8 and standard deviation 1.2.

a Find the z-score for each raw score and describe where it is on the standard normal distribution.

i 25.7 **ii** 20.6 **iii** 28.3

b Which of the scores from part **a** are very unlikely?

c Find the value of a raw score whose z-score is -1.82, correct to 2 decimal places.

Solution

a **i** For $x = 25.7$:

$$z = \frac{x - \mu}{\sigma}$$

$$= \frac{25.7 - 23.8}{1.2}$$

$$\approx 1.58$$

A score of 25.7 lies 1.58 standard deviations above (to the right of) the mean.

ii For $x = 20.6$:

$$z = \frac{x - \mu}{\sigma}$$

$$= \frac{20.6 - 23.8}{1.2}$$

$$\approx -2.67$$

A score of 20.6 lies 2.67 standard deviations below (to the left of) the mean.

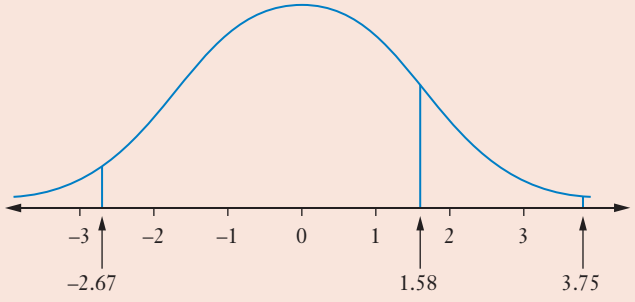
iii For $x = 28.3$:

$$z = \frac{28.3 - 23.8}{1.2}$$

$$\approx 3.75$$

A score of 28.3 lies 3.75 standard deviations above (to the right of) the mean.

b We can draw each score on the standard normal distribution.



99.7% of scores lie within 3 standard deviations of the mean.
So a z-score of 3.75 is very unlikely.

This means that a score of 28.3 is very unlikely.

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c

$$z = \frac{x - \mu}{\sigma}$$

$$-1.82 = \frac{x - 23.8}{1.2}$$

$$-2.184 = x - 23.8$$

$$21.616 = x$$

So the raw score is 21.62.

Exercise 10.07 *z*-scores

1 A data set is normally distributed with mean 18 and standard deviation 1.3.

a Find the z-score for each raw score.

i 18 **ii** 19.3 **iii** 20.6

iv 21.9 **v** 16.7 **vi** 15.4

vii 14.1

b Which raw score has a z-score of:

i 1.5? **ii** –2.1?

2 The length of fish caught in a fishing competition had a mean of 53.1 cm and standard deviation 8.7.

a If the lengths of fish almost certainly lie within 3 standard deviations of the mean, between which lengths would the fish almost certainly lie?

b Find the z-score for each length.

i 53.1 cm **ii** 61.8 cm **iii** 44.4 cm

iv 70.5 cm **v** 35.7 cm **vi** 79.2 cm

vii 27 cm **viii** 65 cm

3 A sample of overnight temperatures at Thredbo in June showed a mean temperature of 6.8°C and a standard deviation of 1.1.

a Almost all temperatures lie within 3 standard deviations of the mean. Within what range do almost all temperatures lie?

b Find the z-score for each temperature.

i 6.8°C **ii** 7.9°C **iii** 9°C

iv 10.1°C **v** 5.7°C **vi** 4.6°C

vii 3.5°C **viii** 6°C

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4 A survey showed that the mean volume of juice in an orange is 66.4 mL with a standard deviation of 5.8.

a The volume of juice very probably lies within 2 standard deviations of the mean. Between which 2 volumes do they lie?

b Find the z-scores for each volume.

i 66.4 mL **ii** 72.2 mL **iii** 78 mL

iv 83.8 mL **v** 60.6 mL **vi** 54.8 mL

vii 49 mL **viii** 90 mL

c Find the volume that has a z-score of:

i 1.2 **ii** 2.9 **iii** –0.6 **iv** –2.3

5 **a** Find the z-score for each raw score below if the mean is 68 and the standard deviation is 4.5.

i 80 **ii** 53.2 **iii** 78.6

iv 62.1 **v** 90 **vi** 59.7

vii 82.7 **viii** 56.4

b From your answers to part **a**, which scores are most unlikely?

c Which scores in part **a** lie within 2 standard deviations of the mean?

d Which scores in part **a** lie within 3 standard deviations of the mean?

6 The mean diameter of a batch of circular discs is 14.2 mm with standard deviation 1.4.

a What is the z-score for a disc with a diameter of :

i 16 mm? **ii** 12 mm?

b Find the diameter of a disc with z-score:

i –2.1 **ii** 1.3 **iii** 3.2

iv –0.76 **v** 1.95

7 A set of data that is normally distributed has a mean of 23 and standard deviation 2. Which raw score has a z-score of 2.5?

8 A data set that is normally distributed has a standard deviation of 4.5. A score of 39 has a z-score of 2.7. What is the mean?

9 Find the standard deviation of a normally distributed data set if the mean is 89 and a raw score of 59 has z-score of –0.6.

10 A set of data that is normally distributed has a mean of 53.4 and standard deviation of 5.6. Find the raw score that has a z-score of:

a 0 **b** –2 **c** 1

d 2.8 **e** –1.7

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11 The standard deviation of a normal distribution is 3.3 and the z-score of 45 is -1. Calculate the mean.

12 The mean of a normally distributed data set is 16 and standard deviation is 1.9.

a Find the scores between which:

i 95% of data lies **ii** 68% of data lies **iii** 99.7% of data lies

b Calculate the z-score of:

i 20 **ii** 13.5

c Find the raw score that has a z-score of:

i -3 **ii** 1.1

13 A normal distribution has a mean of 104.7 and standard deviation 5.1.

a Find the scores that lie within 1 standard deviation of the mean.

b Calculate the z-score of:

i 80 **ii** 103

c Which score has a z-score of:

i 2? **ii** -1.3?

10.08 Applications of the normal distribution

The normal distribution is often used in quality control and predicting outcomes.

EXAMPLE 18

a A company produces 1 kg packets of sugar. A quality control check found that the weight of the packets was normally distributed with a mean weight of 0.995 kg and standard deviation 0.03 kg. The company policy is to reject any packet with a weight outside 2 standard deviations from the mean.

i What is the smallest weight allowed by the company?

ii What is the largest weight allowed?

iii What percentage of packets will be rejected?

iv What percentage of large packets will be rejected?

b The mean shelf life of a spice is 13.4 weeks and the standard deviation is 1.8 weeks.

i Would a shelf life of 20 weeks be unusual? Why?

ii Find the z-score for a shelf life of 15.5 weeks.

iii What percentage of shelf lives would be expected to be between 13.4 and 15.5 weeks?

WS Applying the normal distribution

WS Normal distribution worded problems 1

WS Normal distribution worded problems 2

WS Continuous random variables assignment

WS Continuous random variables problems

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Solution

a $\mu = 0.995$

i Smallest weight allowed:

$\mu - 2\sigma = 0.995 - 2 \times 0.03$

$= 0.935$

ii Largest weight allowed:

$\mu + 2\sigma = 0.995 + 2 \times 0.03$

$= 1.055$

iii About 95% of weights lie within 2 standard deviations of the mean, so 5% will lie outside this area.

The company rejects 5% of the packets of sugar.

iv Since the normal distribution is symmetrical, the 5% is made up of 2.5% of larger and 2.5% of smaller packets.

So the company rejects 2.5% of larger packets.

b **i** $\mu = 13.4$

$\mu - 3\sigma = 13.4 - 3 \times 1.8$

$= 8$

$\mu + 3\sigma = 13.4 + 3 \times 1.8$

$= 18.8$

So the shelf life of spices almost certainly lies between 8 and 18.8 weeks.

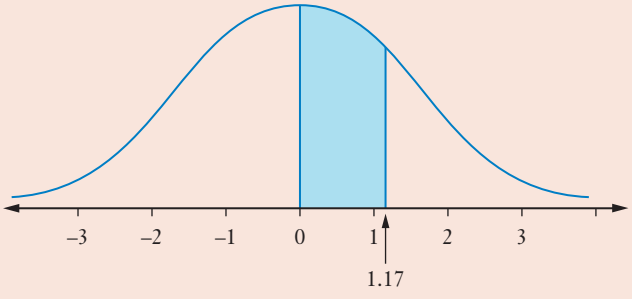
A shelf life of 20 weeks is outside this range, so it would be unusual.

ii $z = \dfrac{x - \mu}{\sigma}$

$= \dfrac{15.5 - 13.4}{1.8}$

$= 1.17$

iii



$\mu = 13.4$ so its z-score $= 0$

$P(13.4 \leq X \leq 15.5) = P(0 \leq z \leq 1.17)$

$= P(X \leq 1.17) - P(X \leq 0)$

$= 0.8790 - 0.5000$ (using the table on pages 441–442)

$= 0.379$

So 37.9% of shelf lives would be expected to be between 13.4 and 15.5 weeks.

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Using z-scores also allows us to compare 2 data sets.

EXAMPLE 19

In Year 7 at a school the mean weight of students was 59.4 kg and the standard deviation was 3.8 kg. In Year 8, the mean was 63.5 kg and the standard deviation was 1.7 kg.

John in Year 7, and Deng in Year 8, both weighed 68 kg. Which student was heavier in relation to his Year?

Solution

For John:

$$z = \frac{x - \mu}{\sigma}$$
$$= \frac{68 - 59.4}{3.8}$$
$$\approx 2.263$$

For Deng:

$$z = \frac{x - \mu}{\sigma}$$
$$= \frac{68 - 63.5}{1.7}$$
$$\approx 2.647$$

A z-score of 2.647 is higher than a z-score of 2.263.

So Deng weighed more in relation to his Year group than John.

Exercise 10.08 Applications of the normal distribution

1 A machine at the mint produces coins with a mean diameter of 24 mm and a standard deviation of 0.2 mm.

a What percentage of coins will have a diameter between:

i 23.4 mm and 24.6 mm?

ii 24 mm and 24.4 mm?

iii 23.8 mm and 24.2 mm?

iv 23.4 mm and 24 mm?

v 23.4 mm and 24.2 mm?

b A coin selected at random has a diameter of 24.8 mm.

i What is its z-score?

ii Is this diameter unusual? What could you say about this?

c **i** Convert a diameter of 23.7 mm to a z-score.

ii Find the probability of producing a coin with a diameter of less than 23.7 mm.

iii Find the probability of producing a coin with a diameter between 23.7 mm and 23.9 mm.

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2 The maximum temperature in April is normally distributed with a mean of 25.3°C and a standard deviation of 3.4°C.

a What percentage of the time in April would you expect the temperature to be between:

i 21.9° and 28.7°? **ii** 18.5° and 32.1°? **iii** 15.1° and 35.5°?

iv 25.3° and 28.7°? **v** 21.9° and 32.1°?

b The temperature drops to 14°.

i What is its z-score?

ii Is this temperature unusual? Why?

c The temperature rises to 38°.

i What is its z-score?

ii Is this temperature unusual? Why?

3 A company surveyed the amount of time it took to assemble its product. The times were normally distributed with mean 8 minutes and standard deviation 1.7 minutes.

a What is the z-score for a time of 15 minutes? Would this be an acceptable amount of time to assemble the product? Why?

b Between what times would 95% of times lie?

c What percentage of times would lie between 4.6 and 13.1 minutes?

d **i** Find the z-score for a time of 7 minutes 30 seconds.

ii What percentage of times would lie between 7 minutes 30 seconds and 8 minutes?

e Find the probability of times lying between:

i 6 and 8 minutes **ii** 6 and 9.7 minutes **iii** 5 and 10 minutes

f What percentage of times would lie between:

i 3 and 12 minutes? **ii** 7 and 11 minutes? **iii** 3.5 and 11.9 minutes?

4 A certain brand of perfume is sold in 20 mL bottles. The volume of perfume in the bottles was tested in a quality control check. The mean was found to be 19.9 mL with a standard deviation of 0.4 mL.

a What percentage of bottles have a volume between:

i 19.1 mL and 20.7 mL?

ii 19.9 mL and 21.1 mL?

iii 20 mL and 21 mL?

b Between which two volumes do 99.7% of bottles lie?

c Comment on a bottle that has a volume of 23 mL.

d Find the probability that a bottle of perfume will have a volume between 18.9 and 19.3 mL.

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5 A farmer is only allowed to deliver standard size apples to the markets. The mean diameter must be 7.5 cm with a standard deviation of 0.3 cm. Only apples within 2 standard deviations of the mean are allowed.

a What percentage of apples are allowed?

b What is the largest diameter allowed?

c What is the smallest diameter allowed?

6 A certain brand of car battery has a mean life of 3.1 years with a standard deviation of 0.3 years.

a What is the minimum life you could reasonably expect from a battery of this type?

b What percentage of batteries would have a life between:

i 2.8 and 3.4 years?

ii 2.5 and 3.5 years?

c Would a life of over 4 years be unusual? Why?

7 A Jack Russell terrier has a mean height of 28 cm and a standard deviation of 0.833 cm.

a What range of heights (to 1 decimal place) would you expect for this breed of dog?

b What is the z-score of a height of 27.2 cm?

c What percentage of dogs would be between 27.2 cm and 30 cm tall?

d Would a dog 24 cm tall be typical of this breed? Why?

8 A factory produces 5 kg bags of bread mix. In a quality check, the mean weight of a bag of bread mix was found to be 4.95 kg, with a standard deviation of 0.15 kg. The factory rejects bags that weigh less than 4.65 kg or more than 5.25 kg.

a What percentage of bags does the factory reject?

b What percentage of bags does the factory reject because their weight is too small?

c The manager of the factory decided that too many bags are being rejected and that in future only those outside the normal range of 3 standard deviations would be rejected.

i What percentage will be rejected?

ii What weights will be rejected?

9 A company manufactures steel rods with a mean diameter of 10.6 cm and a standard deviation of 0.5 cm. The manufacturer rejects rods that are outside acceptable limits.

a If the company rejects 0.3% of rods, find:

i the percentage of rods it accepts

ii the largest diameter it will accept

iii the smallest diameter it will accept

b In one batch of rods, the diameters are 10.9 cm, 9.6 cm, 8.3 cm, 11.4 cm and 12.6 cm. Which ones will be rejected?

10 A manufacturer of canned fruit guarantees that the minimum weight in each can is 250 g. A random check showed that the mean weight was 252.5 g with a standard deviation of 0.4 g. Comment on this guarantee. Is it realistic? Why?

11 A butcher shop advertises that it will give a free leg of lamb to any customer who can prove that a packet of mince with a mean weight of 1 kg weighs less than 980 g. If the mean weight is 1 kg with a standard deviation of 10 g, what percentage of customers should expect to receive a free leg of lamb?

12 The width of a type of door almost certainly lies within the range 814 mm to 826 mm (3 standard deviations of the mean).

- **a** What is the standard deviation?
- **b** What is the mean width of the door?
- **c** Within what widths do 95% of these doors lie?

13 The mean time for a ferry to travel from one port to another is 3.1 hours. About 47.5% of the time the ferry takes between 3.1 hours and 3.5 hours.

- **a** What is the standard deviation?
- **b** What is the minimum time you would expect the ferry to take?

14 Xavier takes 78 minutes to drive to Epping and 65 minutes to drive to the city. The mean time to Epping is 75.3 minutes with a standard deviation of 2.6, while the mean time to the city is 62.7 minutes with a standard deviation of 1.7.

- **a** Calculate the z-scores for Xavier's trips to Epping and the city.
- **b** Which was the longer trip in comparison with the mean?

15 Kieran scored 78 in an exam where the mean was 69.5 and the standard deviation was 8.5. Cameron scored 71 in an exam where the mean was 61.2 and the standard deviation was 4.8.

- **a** Find Kieran's z-score.
- **b** Calculate Cameron's z-score.
- **c** Which student scored higher in comparison with the other students in each exam?

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10. TEST YOURSELF

WS
Formula sheet: Statistics and probability
Qz
Practice quiz

For Questions **1** to **3**, choose the correct answer **A**, **B**, **C** or **D**.

1 Which function does not describe a continuous probability distribution?

A $f(x) = \dfrac{x^2}{21}$ for the interval $1 \leq x \leq 4$

B $f(x) = \dfrac{e^x}{e^3 - 1}$ in the domain $[0, 3]$

C $f(x) = \dfrac{x^4}{625}$ in the domain $[1, 5]$

D $f(x) = \dfrac{4x^3}{625}$ in the interval $0 \leq x \leq 5$

2 The percentage of z-scores between -2 and 2 is:

A 95% **B** 68% **C** 47.5% **D** 49.85%

3 Which random variable is not continuous?

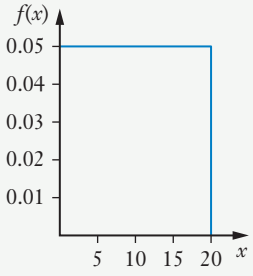
A The temperature of different freezers

B The mass of rocks found at the site of a volcano

C The length of the arms of people

D Shoe sizes of people

4 For the uniform continuous probability distribution shown, find:



a $P(X \leq 15)$ **b** $P(X \leq 8)$

c $P(7 \leq X \leq 18)$ **d** $P(4 < X < 13)$

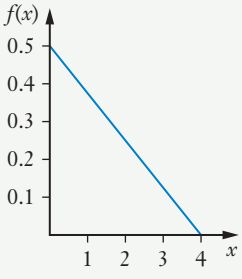
e $P(X \geq 6)$ **f** the median

g the 18th percentile **h** the 89th percentile

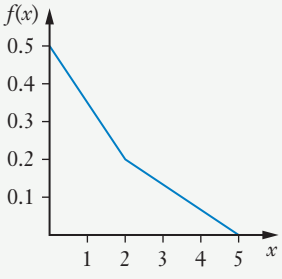
i the 6th decile **j** the 3rd quartile

5 State whether each graph represents a probability density function.

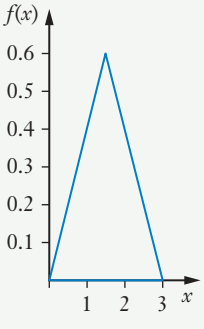
a



b



c



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6 Find the median of the continuous random variable $f(x) = \frac{3x^2}{124}$ defined on the interval $1 \leq x \leq 5$. Answer correct to 2 decimal places.

7 **a** Find the cumulative distribution function for $f(x) = \begin{cases} \frac{3x^2}{511} & \text{for } 1 \leq x \leq 8 \\ 0 & \text{for all other } x \end{cases}$

b Find:

i $P(X \leq 3)$ **ii** $P(X \leq 5)$ **iii** $P(X > 6)$

iv $P(X \geq 4)$ **v** $P(2 \leq X \leq 7)$

8 For each continuous probability distribution, find:

i the mode **ii** the median

a $f(x) = \frac{2x}{15}$ defined in the domain [1, 4]

b $f(x) = \frac{x^2}{243}$ defined in the interval $0 \leq x \leq 9$

9 The birth weights of babies born at St John's Hospital were measured and found to be normally distributed with mean 3.2 kg and standard deviation 0.31.

a Find the range of weights in which 99.7% of the weights of these babies would lie.

b Find the z-score (to 2 decimal places) for a weight of:

i 3.9 kg **ii** 3.5 kg

c Use the standard normal probability table on pages 441–442 to find the probability that a baby born at the hospital would have a birth weight between 3.5 kg and 3.9 kg.

10 A function is given by $f(x) = \frac{3x^5}{2048}$. Over what domain starting at $x = 0$ is this a probability density function?

11 Draw a probability density function for the normal distribution with:

a mean 15 and standard deviation 0.5

b mean 3.4 and standard deviation 0.2

12 A factory produces 2 kg bags of nails. In a quality check, the mean weight of a bag of nails was found to be 1.95 kg, with a standard deviation of 0.08 kg. The factory rejects bags that weigh less than 1.71 kg or more than 2.19 kg.

a What percentage of bags does the factory reject?

b After a complaint, the manager decided to reject bags that weigh less than 1.87 kg or more than 2.11 kg. What percentage will be rejected?

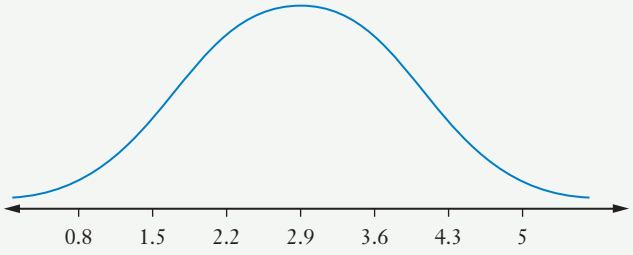
13 For the PDF $f(x) = \frac{3x^2}{316}$ defined on the interval $3 \leq x \leq 7$, find:

a the median **b** the 3rd quartile **c** the 4th decile

d the 63rd percentile **e** the 28th percentile

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14 Find the mean and standard deviation of this normal distribution.



15 Find the cumulative distribution function for each continuous probability distribution.

a $f(x) = \frac{x^4}{625}$ defined on the interval $0 \leq x \leq 5$

b $f(x) = \frac{x^6}{117\,649}$ defined on $[0, 7]$

c $f(x) = \frac{e^x}{e^6 - 1}$ in the interval $0 \leq x \leq 6$

d $f(x) = \frac{x}{40}$ for $[1, 9]$

16 Use the probability table for a standard normal distribution on pages 441–442 to find:

a $P(z \leq 0.54)$ **b** $P(z \leq 1.32)$ **c** $P(z \leq -3)$

d $P(z \leq -0.71)$ **e** $P(z \geq -1)$ **f** $P(z \geq 2.5)$

g $P(z \geq -1.08)$ **h** $P(-2.3 \leq z \leq -1.09)$ **i** $P(1.1 \leq z \leq 3.11)$

17 A set of data is normally distributed with mean 12.5 and standard deviation 1.5.
Find the percentage of data that lies between:

a 9.5 and 15.5 **b** 12.5 and 14 **c** 11 and 17

d 10 and 15 **e** 12 and 13

18 A probability density function is given by $f(x) = ae^x$ over a certain domain.
Find the exact value of a if the domain is:

a $[0, 5]$ **b** $[1, 4]$

19 Shade on the standard normal distribution the area where values lie:

a in the bottom 20%

b in the bottom 32%

c in the top 15%

d in the top 30%

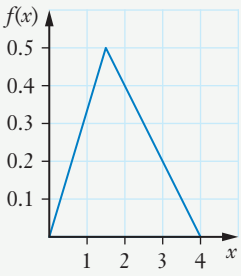
e between the bottom 10% and the top 40%

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20 Find the mode of each continuous probability distribution.

a $f(x) = \frac{x}{30}$ defined in the domain $[2, 8]$

b



c $f(x) = \frac{4}{189}(x^3 - 9x^2 + 24x)$ defined in the domain $[0, 3]$

21 Klare took 9.3 s to finish a race where the mean was 8.3 s and the standard deviation was 1.2. Simon took 8.9 s to run in the next heat where the mean was 8.1 s and the standard deviation was 0.8.

a Find Klare's z-score.

b Calculate Simon's z-score.

c Which person had the better time in comparison with the other runners in their heat?

22 Show that $f(x) = \frac{x^3}{600}$ defined on the interval $1 \leq x \leq 7$ is a probability density function.

23 Circular tables are made with a mean radius of 1.1 m with standard deviation 0.02 m.

a Find the z-score for a table with a radius of:

i 1.15 m **ii** 1.07 m

b Find the radius of a table with z-score:

i 1 **ii** -2 **iii** 3.1 **iv** -0.63 **v** 1.27

24 A normal distribution has a standard deviation of 1.6. A score of 87.9 has a z-score of -1.3. What is the mean?

25 The standard deviation of a normal distribution is 1.9 and the z-score of 52.4 is 1.7. Calculate the mean.

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10. CHALLENGE EXERCISE

1 A probability density function is given by $f(x) = \dfrac{5x^4}{3124}$ over the interval $a \leq x \leq b$.

Find the value of a and b if the median of the PDF is 4.353 031.

2 A normal distribution has mean μ and standard deviation σ.
Evaluate μ and σ given that 95% of scores lie between 12.4 and 14.

3 Find the mode of the continuous probability distribution $f(x) = \dfrac{4}{249}(x + 2)(x - 4)^2$ defined in the domain [0, 3].

4 Find the cumulative distribution function for the continuous probability distribution $f(x) = \dfrac{3x(x^2 + 1)^2}{62\,000}$ defined on the domain [3, 7].

5 For a normal distribution, $P(X \leq 23.8) = 91.92\%$ and $P(X \leq 17.15) = 30.85\%$.
Find the mean and standard deviation.

6 A good model for a normal distribution is the function $f(x) = \dfrac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$.

Use technology to show that this is a probability density function given $\mu = 6$ and $\sigma = 1$.

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