CALCULUS

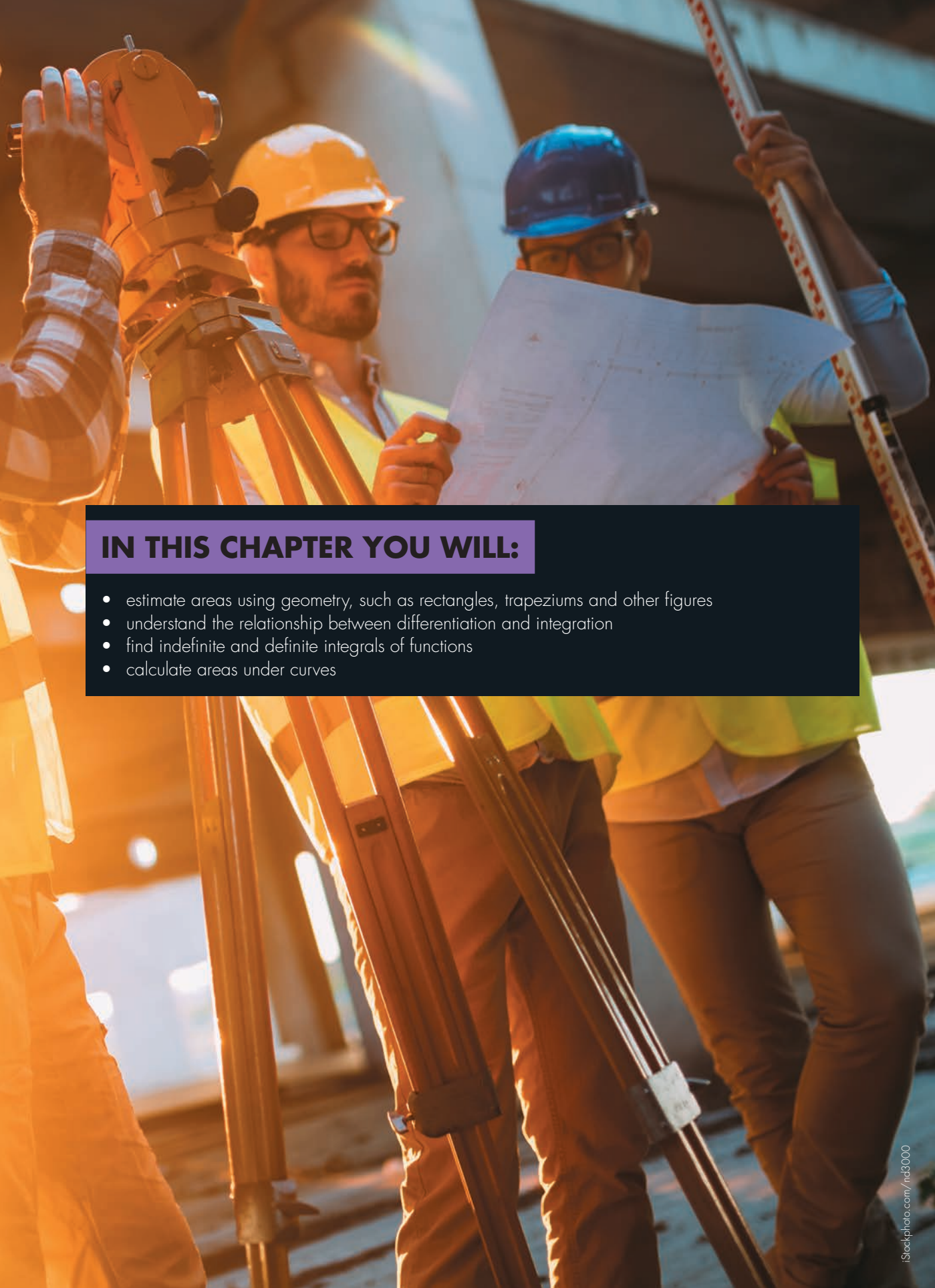
6.

INTEGRATION

Integration is the process of finding an area under a curve. This is used in many areas of knowledge, such as surveying, physics and the social sciences. In this chapter, you will look at how to find both approximate and exact areas under a curve and you will learn how integration and differentiation are related.

CHAPTER OUTLINE

6.01 Approximating areas under a curve
6.02 Trapezoidal rule
6.03 Definite integrals
6.04 Indefinite integrals
6.05 Chain rule
6.06 Integration involving exponential functions
6.07 Integration involving logarithmic functions
6.08 Integration involving trigonometric functions
6.09 Areas enclosed by the *x*-axis
6.10 Areas enclosed by the *y*-axis
6.11 Sums and differences of areas



IN THIS CHAPTER YOU WILL:

- estimate areas using geometry, such as rectangles, trapeziums and other figures
- understand the relationship between differentiation and integration
- find indefinite and definite integrals of functions
- calculate areas under curves

iStockphoto.com/nd3000

TERMINOLOGY

definite integral: The integral or anti-derivative $y = F(x)$ used to find the area between the curve $y = f(x)$, the x-axis and boundaries $x = a$ and $x = b$, given by $\int_a^b f(x)\,dx = F(b) - F(a)$.

indefinite integral: A general anti-derivative $\int f(x)\,dx$.

integral: An anti-derivative.

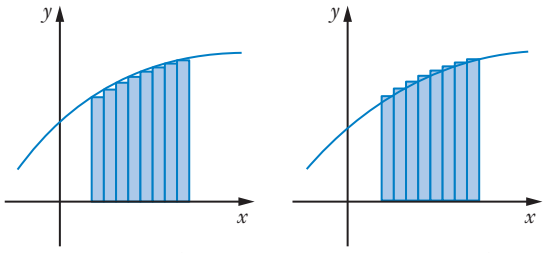
integration: The process of finding an anti-derivative.

trapezoidal rule: A formula for approximating area under a curve by using a trapezium.

WS
Areas using rectangles

6.01 Approximating areas under a curve

Mathematicians since the time of Archimedes have used rectangles to approximate irregular areas. In more recent times, we use the number plane to find areas enclosed between a curve and the x-axis. We call this the **area under the curve**.



inner rectangles outer rectangles

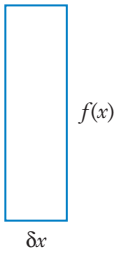
The first diagram has inner or left rectangles that are below the curve, because the top left corners of the rectangles touch the curve.

The second diagram has outer or right rectangles that are above the curve, because the top right corners of the rectangles touch the curve.

The more rectangles we have, the more accurately they approximate the area under the curve.

Integral notation

The diagram at right shows one of the rectangles. The height of each rectangle is $f(x)$ and its width is δx, so its area is $f(x)\,\delta x$. So the sum of all the rectangles is $\Sigma f(x)\,\delta x$ for the different values of x.



We can approximate the area under the curve using a large number of rectangles by making the width of each rectangle very small.

Taking an infinite number of rectangles, $\delta x \to 0$.

$$\text{Area} = \lim_{\delta x \to 0} \left(\sum f(x) \delta x \right)$$
$$= \int f(x)\,dx$$

δx is 'delta x' and means a small change in x. δ is the Greek letter for 'd', for difference.

We use the **integral** symbol $\int$ to stand for the sum of rectangles (the symbol is an S for sum).

We call $\int f(x)\,dx$ an **indefinite integral**.

If we are finding the area under the curve $y = f(x)$ between $x = a$ and $x = b$, we can write $\int_a^b f(x)\,dx$.

We call $\int_a^b f(x)\,dx$ a **definite integral**.

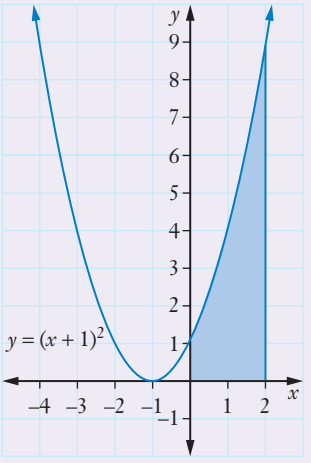
ISBN 9780170413220

EXAMPLE 1

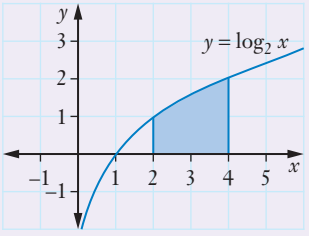
a Find an approximation to the shaded area by using:

i 4 inner rectangles

ii 4 outer rectangles

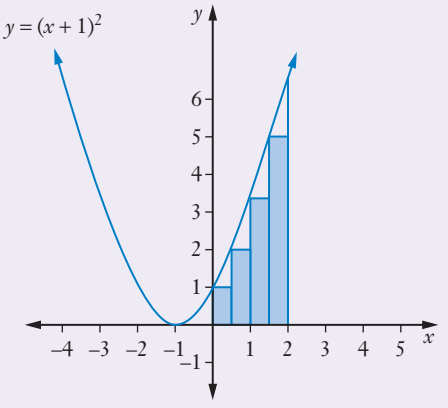


b Find the shaded area below by using a trapezium.



Solution

a **i** Using inner rectangles, the top left corners touch the curve and they lie below the curve.



Each rectangle has height $f(x)$ and width 0.5 units.

Height of 1st rectangle:

$f(0) = (0 + 1)^2 = 1$

Area $= 1 \times 0.5 = 0.5$

Height of 2nd rectangle:

$f(0.5) = (0.5 + 1)^2 = 2.25$

Area $= 2.25 \times 0.5 = 1.125$

Height of 3rd rectangle:

$f(1) = (1 + 1)^2 = 4$

Area $= 4 \times 0.5 = 2$

Height of 4th rectangle:

$f(1.5) = (1.5 + 1)^2 = 6.25$

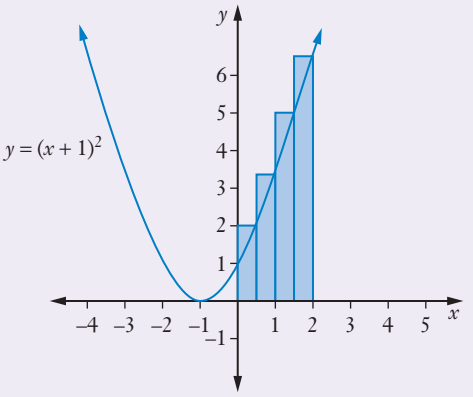
Area $= 6.25 \times 0.5 = 3.125$

Total area $= 0.5 + 1.125 + 2 + 3.125 = 6.75$

So area is 6.75 units^2.

ISBN 9780170413220

ii Using outer rectangles, the top right corners touch the curve and they go above the curve.



Height of 1st rectangle:

$f(0.5) = (0.5 + 1)^2 = 2.25$

Area $= 2.25 \times 0.5 = 1.125$

Height of 2nd rectangle:

$f(1) = (1 + 1)^2 = 4$

Area $= 4 \times 0.5 = 2$

Height of 3rd rectangle:

$f(1.5) = (1.5 + 1)^2 = 6.25$

Area $= 6.25 \times 0.5 = 3.125$

Height of 4th rectangle:

$f(2) = (2 + 1)^2 = 9$

Area $= 9 \times 0.5 = 4.5$

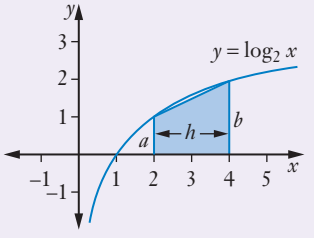
Total area $= 1.125 + 2 + 3.125 + 4.5 = 10.75$

So area is 10.75 units^2

b Area of a trapezium:

$A = \frac{1}{2}h(a + b)$ where h = perpendicular height and a and b are parallel sides.

Here the trapezium has parallel sides at $x = 2$ and $x = 4$.



$a = f(2)$

$= \log_2 2$

$= 1$

$b = f(4)$

$= \log_2 4$

$= 2$

$h = 4 - 2$

$= 2$

$A = \frac{1}{2}h(a + b)$

$= \frac{1}{2}(2)(1 + 2)$

$= 3$

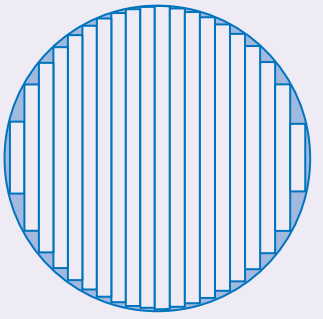
So area is 3 units^2.

ISBN 9780170413220

DID YOU KNOW?

Archimedes

Integration has been of interest to mathematicians since very early times. **Archimedes** (287–212 BCE) found the area of enclosed curves by cutting them into very thin layers and finding their sum. He found the formula for the volume of a sphere this way. He also found an estimation of π, correct to 2 decimal places.



Area within curve



Alamy Stock Photo/Heritage Image Partnership Ltd

Archimedes

TECHNOLOGY

Areas under a curve

We can use a spreadsheet to find approximate areas under a curve using rectangles. Using technology allows us to find sums of large numbers of rectangles without needing to do many calculations. This gives a more accurate approximation to the area under a curve.

For example, we can use a spreadsheet to find the approximate area under the curve $y = (x + 1)^2$ between $x = 0$ and $x = 2$ from Example 1 **a i**.

We find the y values using the formula **=(A2+1)^2** (copy the formula down the column).

The width is **=A3–A2** (copy this value down the column).

The area is **=B2*C2** (copy the formula down the column).

	A	B	C	D
1	x	y	Width	Area
2	0	1	0.5	0.5
3	0.5	2.25	0.5	1.125
4	1	4	0.5	2
5	1.5	6.25	0.5	3.125
6				
7			Total area	6.75

ISBN 9780170413220

We can use the spreadsheet to find the area using a much larger number of rectangles, for example, 20 or 40.

	A	B	C	D	E	F	G	H	I
1	x	y	Width	Area		x	y	Width	Area
2	0	1	0.1	0.1		0	1	0.05	0.05
3	0.1	1.21	0.1	0.121		0.05	1.1025	0.05	0.055125
4	0.2	1.44	0.1	0.144		0.1	1.21	0.05	0.0605
5	0.3	1.69	0.1	0.169		0.15	1.3225	0.05	0.066125
6	0.4	1.96	0.1	0.196		0.2	1.44	0.05	0.072
7	0.5	2.25	0.1	0.225		0.25	1.5625	0.05	0.078125
8	0.6	2.56	0.1	0.256		0.3	1.69	0.05	0.0845
9	0.7	2.89	0.1	0.289		0.35	1.8225	0.05	0.091125
10	0.8	3.24	0.1	0.324		0.4	1.96	0.05	0.098
11	0.9	3.61	0.1	0.361		0.45	2.1025	0.05	0.105125
12	1	4	0.1	0.4		0.5	2.25	0.05	0.1125
13	1.1	4.41	0.1	0.441		0.55	2.4025	0.05	0.120125
14	1.2	4.84	0.1	0.484		0.6	2.56	0.05	0.128
15	1.3	5.29	0.1	0.529		0.65	2.7225	0.05	0.136125
16	1.4	5.76	0.1	0.576		0.7	2.89	0.05	0.1445
17	1.5	6.25	0.1	0.625		0.75	3.0625	0.05	0.153125
18	1.6	6.76	0.1	0.676		0.8	3.24	0.05	0.162
19	1.7	7.29	0.1	0.729		0.85	3.4225	0.05	0.171125
20	1.8	7.84	0.1	0.784		0.9	3.61	0.05	0.1805
21	1.9	8.41	0.1	0.841		0.95	3.8025	0.05	0.190125
22	2	9	0.1	0.9		1	4	0.05	0.2
23						1.05	4.2025	0.05	0.210125
24						1.1	4.41	0.05	0.2205
25			Total area	9.17		1.15	4.6225	0.05	0.231125
26						1.2	4.84	0.05	0.242
27						1.25	5.0625	0.05	0.253125
28						1.3	5.29	0.05	0.2645
29						1.35	5.5225	0.05	0.276125
30						1.4	5.76	0.05	0.288
31						1.45	6.0025	0.05	0.300125
32						1.5	6.25	0.05	0.3125
33						1.55	6.5025	0.05	0.325125
34						1.6	6.76	0.05	0.338
35						1.65	7.0225	0.05	0.351125
36						1.7	7.29	0.05	0.3645
37						1.75	7.5625	0.05	0.378125
38						1.8	7.84	0.05	0.392
39						1.85	8.1225	0.05	0.406125
40						1.9	8.41	0.05	0.4205
41						1.95	8.7025	0.05	0.435125
42						2	9	0.05	0.45
43									
44								Total area	8.9175
45									

Use technology with a larger number of rectangles for a more accurate area to this question.

ISBN 9780170413220

We can find the area under the same curve by using different methods.

We can use other shapes to find areas under a curve.

EXAMPLE 2

Find an approximation to the area under the curve $y = x^2$ between $x = 0$ and $x = 2$ by using:

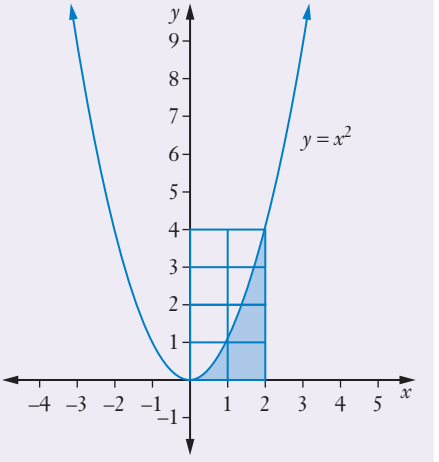
a squares **b** a triangle

Solution

a On the grid, each square is 1 square unit.
By counting and approximating squares:

$A \approx 3$

So area is 3 units^2.



b Using a triangle:

$b = 2 - 0.5 = 1.5$

$h = f(2)$

$= 2^2$

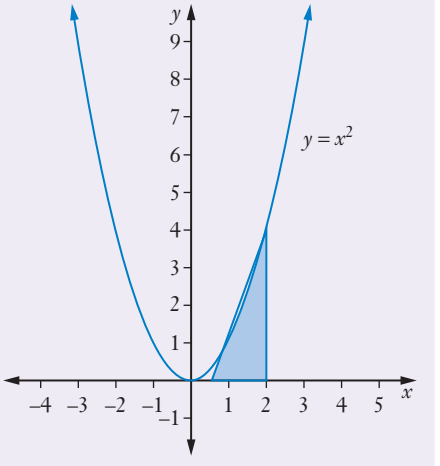
$= 4$

$A = \frac{1}{2}bh$

$= \frac{1}{2} \times 1.5 \times 4$

$= 3$

So area is 3 units^2.



ISBN 9780170413220

Exercise 6.01 Approximating areas under a curve

1 Find an approximation to the area under the curve $y = x^2 + 2x$ between $x = 1$ and $x = 2$ by using:

a 2 inner rectangles **b** 2 outer rectangles

2 Find an approximation (to 2 decimal places) to the area under the curve $y = \dfrac{2}{x+1}$ from $x = 1$ to $x = 3$ using:

a 2 inner rectangles **b** 2 outer rectangles

c 4 inner rectangles **d** 4 outer rectangles

3 Use a trapezium to find an approximate area under the curve:

a $f(x) = x^2$ between $x = 2$ and $x = 3$

b $y = \ln x$ between $x = 4$ and $x = 7$

c $f(x) = x^3 + 1$ between $x = 0$ and $x = 4$

d $f(x) = \sin x$ between $x = \dfrac{\pi}{4}$ and $x = \dfrac{\pi}{2}$ (give answer in exact form)

e $y = 9 - x^2$ between $x = 1$ and $x = 2$

4 Find the approximate area under the curve $f(x) = x^3 + 3$ between $x = 0$ and $x = 4$ by using:

a 2 inner rectangles **b** 2 outer rectangles

c a trapezium

5 Use a trapezium to find each area under the curve.

a $y = \dfrac{1}{x}$ between $x = 1$ and $x = 7$

b $y = x^2 + 5$ between $x = 0$ and $x = 1$

c $f(x) = \cos x$ between $x = 0$ and $x = \dfrac{\pi}{3}$ (in exact form)

d $y = e^x$ between $x = 1$ and $x = 4$ (in exact form)

e $f(x) = x(x - 4)(x - 9)$ between $x = 2$ and $x = 3$

6 **a** Sketch the graph of $y = 1 - x^2$ and shade the area under the curve (enclosed between the curve and the x-axis).

b Find this approximate area by using a triangle.

7 Find the approximate area under the curve $y = \sqrt{x - 1}$ between $x = 2$ and $x = 5$ by using:

a 6 inner rectangles **b** 6 outer rectangles

c a trapezium **d** squares

8 Find the exact area under the curve $y = \sqrt{25 - x^2}$.

ISBN 9780170413220

9 **a** Find the exact area under the curve $y = \sqrt{9 - x^2}$.

b Find the approximate area under the curve $y = \sqrt{9 - x^2}$:

i between $x = 1$ and $x = 2$ using a trapezium

ii between $x = 0$ and 3 using 3 outer rectangles

10 Use a triangle to find the approximate area under the curve:

a $y = x^2$ between $x = 0$ and $x = 4$

b $y = \sqrt{x}$ between $x = 0$ and $x = 3$

c $y = \cos x$ between $x = 0$ and $x = \dfrac{\pi}{2}$

11 Find the approximate area under each curve by using:

i 4 inner rectangles **ii** 4 outer rectangles

a $y = -x^2 + 4x$

b $y = \sin x$ in the domain $[0, \pi]$ (in exact form)

12 Find the approximate area under the curve $y = x^2 + 5$ between $x = 0$ and $x = 5$ (using technology where available) using:

a 10 inner rectangles

b 10 outer rectangles

6.02 Trapezoidal rule

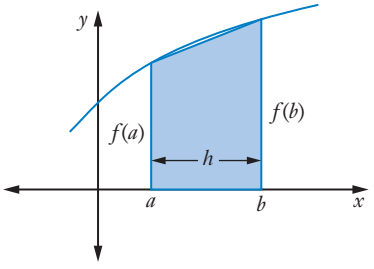
WS Trapezoidal rule

A trapezium usually gives a much closer approximation to the area under a curve than a rectangle does.

The **trapezoidal rule** is a formula that uses a trapezium to find the area under a curve.

$$A = \frac{1}{2} h[f(a) + f(b)] \text{ where } h = b - a$$

$$= \frac{1}{2} (b - a)[f(a) + f(b)]$$



Trapezoidal rule

$$\int_a^b f(x)\,dx \approx \frac{1}{2} (b - a)[f(a) + f(b)]$$

ISBN 9780170413220

EXAMPLE 3

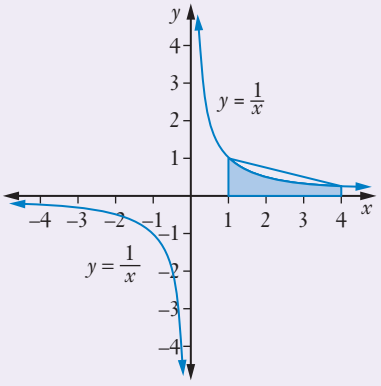
Use the trapezoidal rule to find an approximation for:

a $\int_1^4 \frac{1}{x}\,dx$

b $\int_0^1 x^3\,dx$ using 2 subintervals

Solution

a $\int_1^4 \frac{1}{x}\,dx$ is the area under the curve as shaded in the diagram.



$f(x) = \frac{1}{x}$, $a = 1$ and $b = 4$.

$$\int_a^b f(x)\,dx \approx \frac{1}{2}(b-a)[f(a)+f(b)]$$

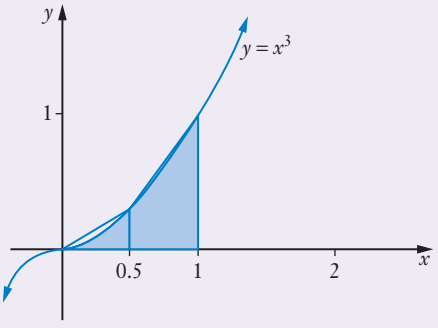
$$\int_1^4 \frac{1}{x}\,dx \approx \frac{1}{2}(4-1)[f(1)+f(4)]$$

$$= \frac{1}{2}(3)\left[\frac{1}{1}+\frac{1}{4}\right]$$

$$= \frac{15}{8}$$

$$= 1\frac{7}{8}$$

b 2 subintervals means 2 trapezia.
We use the trapezoidal formula twice.



$f(x) = x^3$ and $h = 0.5$

$$\int_a^b f(x)\,dx \approx \frac{1}{2}(b-a)[f(a)+f(b)]$$

$$\int_0^1 x^3\,dx = \int_0^{0.5} x^3\,dx + \int_{0.5}^1 x^3\,dx$$

$$\approx \frac{1}{2}(0.5-0)[f(0)+f(0.5)] + \frac{1}{2}(1-0.5)[f(0.5)+f(1)]$$

$$= \frac{1}{2}(0.5)[0^3+0.5^3] + \frac{1}{2}(0.5)[0.5^3+1^3]$$

$$= 0.3125$$

ISBN 9780170413220

There is a more general trapezoidal rule when using several subintervals or trapezia.

Trapezoidal rule for *n* subintervals

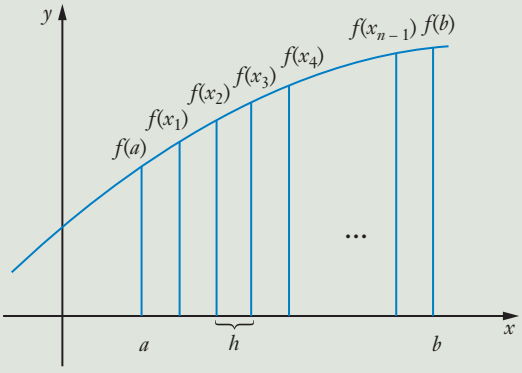
Given n subintervals (trapezia)

$$\int_a^b f(x)\,dx \approx \frac{h}{2}\left[f(a)+f(b)+2\left\{f(x_1)+\ldots+f(x_{n-1})\right\}\right]$$

where $a = x_0$ and $b = x_n$, and the values of $x_0, x_1, x_2, \ldots, x_n$ are found by dividing the interval $a \le x \le b$ into n equal subintervals of width $h = \dfrac{b-a}{n}$.

Since $h = \dfrac{b-a}{n}$, the formula can also be written as

$$\int_a^b f(x)\,dx \approx \frac{b-a}{2n}\left[f(a)+f(b)+2\left\{f(x_1)+\ldots+f(x_{n-1})\right\}\right].$$



Proof

Interval $b - a$ is divided into n trapezia.

So the width of each trapezium is $h = \dfrac{b-a}{n}$.

$$\int_a^b f(x)\,dx \approx \frac{1}{2}h[f(a)+f(x_1)] + \frac{1}{2}h[f(x_1)+f(x_2)] + \frac{1}{2}h[f(x_2)+f(x_3)] + \ldots + \frac{1}{2}h[f(x_{n-1})+f(b)]$$

$$= \frac{h}{2}[f(a)+f(x_1)+f(x_1)+f(x_2)+f(x_2)+f(x_3)+\ldots+f(x_{n-1})+f(b)]$$

$$= \frac{h}{2}\left(f(a)+2f(x_1)+2f(x_2)+2f(x_3)+\ldots+2f(x_{n-1})+f(b)\right)$$

$$= \frac{h}{2}\left[f(a)+f(b)+2\left\{f(x_1)+\ldots+f(x_{n-1})\right\}\right]$$

ISBN 9780170413220

EXAMPLE 4

Trapezoidal rule

a Use the trapezoidal rule with 4 subintervals to find an approximation for $\int_2^3 \frac{2}{x-1}\,dx$ correct to 3 decimal places.

b Use the trapezoidal rule with 7 subintervals to find an approximation for $\int_0^{14} (t^2+3)\,dt$.

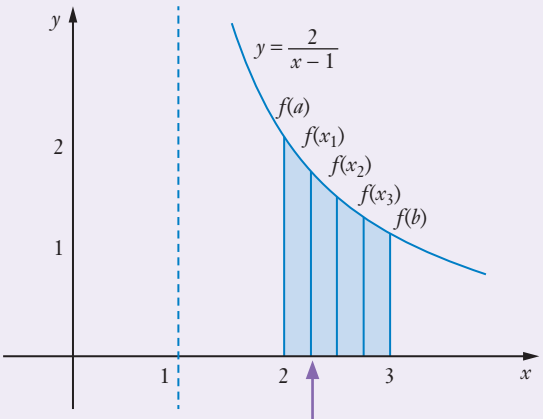
Solution

a From the question and diagram:

$a = 2, b = 3, n = 4.$

The width of each trapezium is

$$h = \frac{b-a}{n}$$
$$= \frac{3-2}{4}$$
$$= \frac{1}{4}$$
$$= 0.25$$



Substituting into the general trapezoidal rule:

$$\int_a^b f(x)\,dx \approx \frac{h}{2}\left[f(a)+f(b)+2\{f(x_1)+\ldots+f(x_{n-1})\}\right]$$

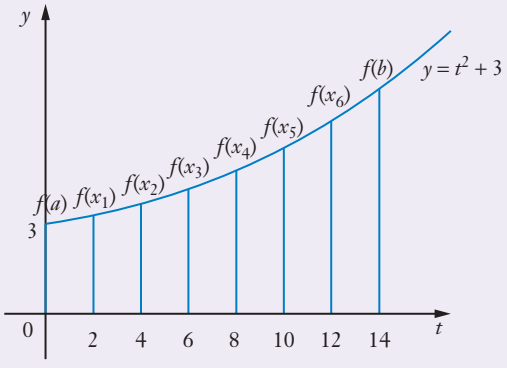
Note that these x values increase by $h = 0.25$, the width of each trapezium.

$$\int_2^3 \frac{2}{x-1}\,dx \approx \frac{0.25}{2}\left[f(2)+f(3)+2\{f(2.25)+f(2.5)+f(2.75)\}\right]$$
$$= 0.125\left[\frac{2}{2-1}+\frac{2}{3-1}+2\left(\frac{2}{2.25-1}+\frac{2}{2.5-1}+\frac{2}{2.75-1}\right)\right]$$
$$= 0.125[2 + 1 + 2(1.6 + 1.3333\ldots + 1.1428\ldots)]$$
$$= 1.39404\ldots$$
$$\approx 1.394$$

b $a = 0, b = 14, n = 7.$

The width of each trapezium is

$$h = \frac{14-0}{7}$$
$$= \frac{14}{7}$$
$$= 2$$



ISBN 9780170413220

Substituting into the general trapezoidal rule:

$$\int_a^b f(x)\,dx \approx \frac{h}{2}\left[f(a)+f(b)+2\left\{f(x_1)+\ldots+f(x_{n-1})\right\}\right]$$

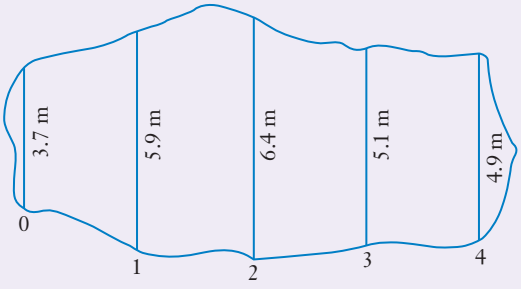
$$\begin{aligned}\int_0^{14}(t^2+3)\,dt &\approx \frac{2}{2}\left[f(0)+f(14)+2\left\{f(2)+f(4)+f(6)+f(8)+f(10)+f(12)\right\}\right]\\ &= [(0^2+3)+(14^2+3)+2\{(2^2+3)+(4^2+3)+(6^2+3)+(8^2+3)\\ &\quad +(10^2+3)+(12^2+3)\}]\\ &= 966\end{aligned}$$

We can use the trapezoidal rule to find irregular areas.

EXAMPLE 5

A surveyor needs to find the area of the irregular piece of land shown.

Use the trapezoidal rule to find its approximate area.



Solution

We can use the values in the diagram or put them in the table below.

x	0	1	2	3	4
$f(x)$	3.7	5.9	6.4	5.1	4.9

$a = 0, b = 4, n = 4$.

From the diagram or table, the height of each trapezium is 1.

$$\begin{aligned}\text{Area} &\approx \frac{h}{2}\left[f(a)+f(b)+2\left\{f(x_1)+\ldots+f(x_{n-1})\right\}\right]\\ &= \frac{1}{2}\left[f(0)+f(4)+2\left\{f(1)+f(2)+f(3)\right\}\right]\\ &= \frac{1}{2}\left[3.7+4.9+2\left\{5.9+6.4+5.1\right\}\right]\\ &= 21.7\end{aligned}$$

So the area of the land is approximately 21.7 m^2.

ISBN 9780170413220

Exercise 6.02 Trapezoidal rule

1 Use the trapezoidal rule to find an approximation for each integral.

a $\int_1^2 x^2\,dx$ **b** $\int_0^2 (x^3+1)\,dx$ **c** $\int_1^5 \frac{dx}{x}$ **d** $\int_1^2 \frac{dx}{x+3}$

2 Find an approximation to $\int_1^3 x^3\,dx$ using the trapezoidal rule with:

a 1 subinterval **b** 2 subintervals

3 Use the trapezoidal rule with 2 trapezia to find an approximation to:

a $\int_2^3 \log x\,dx$ **b** $\int_0^2 \frac{dx}{x+4}$

4 Find an approximation to:

a $\int_1^4 \log x\,dx$ using 3 trapezia **b** $\int_0^2 (x^2-x)\,dx$ using 4 trapezia

c $\int_0^1 \sqrt{x}\,dx$ using 5 subintervals **d** $\int_1^5 \frac{dx}{x^2}$ using 4 subintervals

e $\int_3^6 \frac{dx}{x-1}$ using 6 trapezia

5 Given the table of values, find the approximate value of each definite integral.

a $\int_1^9 f(x)\,dx$

x	1	3	5	7	9
$f(x)$	3.2	5.9	8.4	11.6	20.1

b $\int_1^4 f(t)\,dt$

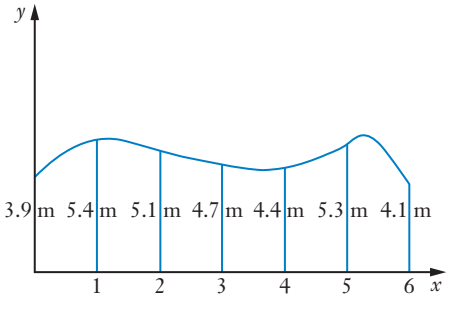
t	1	2	3	4
$f(t)$	8.9	6.5	4.1	2.9

c $\int_2^{14} f(x)\,dx$

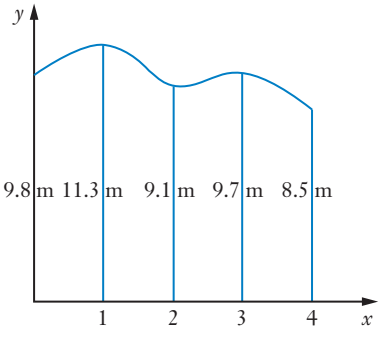
x	2	4	6	8	10	12	14
$f(x)$	25.1	37.8	52.3	89.3	67.8	45.4	39.9

6 Use the trapezoidal rule to find the approximate area of each irregular figure below.

a

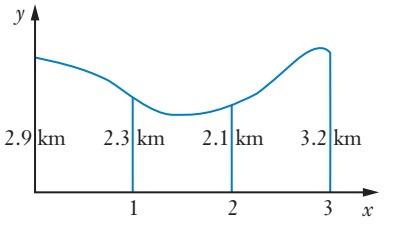


b

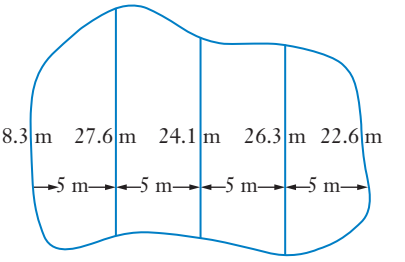


ISBN 9780170413220

c



d



6.03 Definite integrals

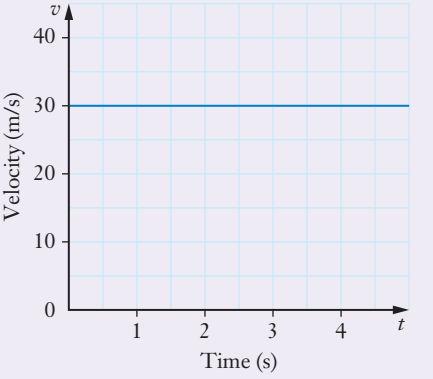
We can link the area under a graph to calculus.

WS Definite integrals

WS Finding definite integrals

EXAMPLE 6

This graph shows the velocity of an object over time as it travels at a constant 30 m s^{-1}.



a Find the distance it travels in:

- **i** 1 s
- **ii** 2 s
- **iii** 3 s

b Find the area under the line between:

- **i** $t = 0$ and $t = 1$
- **ii** $t = 0$ and $t = 2$
- **iii** $t = 0$ and $t = 3$

Solution

a **i** $s = \frac{d}{t}$, so $d = st$.

In 1 s, the object travels $30 \times 1 = 30$ m.

ii In 2 s, the object travels $30 \times 2 = 60$ m.

iii In 3 s, the object travels $30 \times 3 = 90$ m.

b **i** The area is $30 \times 1 = 30$ units^2

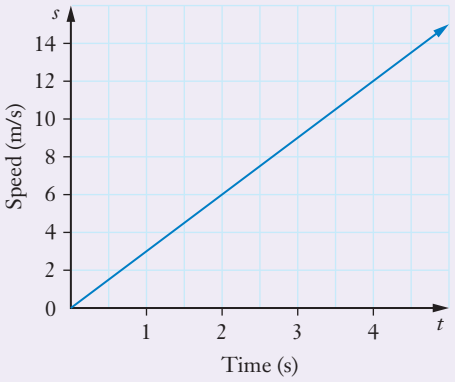
ii The area is $30 \times 2 = 60$ units^2

iii The area is $30 \times 3 = 90$ units^2

ISBN 9780170413220

EXAMPLE 7

This graph shows the speed of an object increasing at a steady rate.



a Find the distance travelled in:

i 1 s

ii 4 s

b Find the area under the graph between:

i $t = 0$ and $t = 1$

ii $t = 0$ and $t = 4$

Solution

a **i** At $t = 0$, $s = 0$, and at $t = 1$, $s = 3$.

$$\text{Average speed} = \frac{0+3}{2}$$
$$= 1.5 \text{ m s}^{-1}$$

In 1 s, the object travels
$1.5 \times 1 = 1.5$ m.

ii At $t = 0$, $s = 0$, and at $t = 4$, $s = 12$.

$$\text{Average speed} = \frac{0+12}{2}$$
$$= 6 \text{ m s}^{-1}$$

In 4 s, the object travels
$6 \times 4 = 24$ m.

b **i** $A = \frac{1}{2}bh$

$$= \frac{1}{2} \times 1 \times 3$$
$$= 1.5 \text{ units}^2$$

ii $A = \frac{1}{2}bh$

$$= \frac{1}{2} \times 4 \times 12$$
$$= 24 \text{ units}^2$$

The graphs in the last 2 examples show velocity (rate of change of displacement) and speed (rate of change of distance) against time. The area under each curve gave the information about the original variable. This is the anti-derivative.

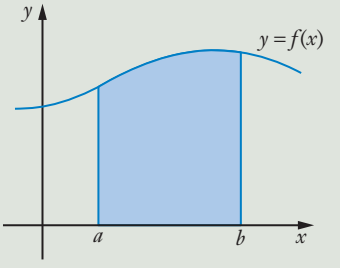
In the same way, the area under any rate of change graph will give the original variable, or the anti-derivative.

Fundamental theorem of calculus

The area enclosed by the curve $y = f(x)$, the x-axis and the lines $x = a$ and $x = b$ is given by

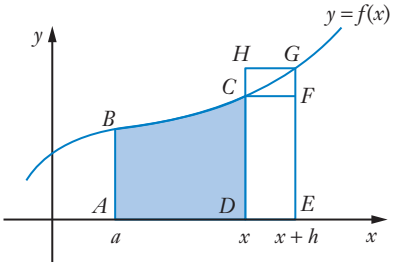
$$\int_a^b f(x)\,dx = F(b) - F(a)$$

where $F(x)$ is the anti-derivative of function $f(x)$.



ISBN 9780170413220

Proof



Consider a continuous curve $y = f(x)$ for all values of $x > a$.

Let area $ABCD$ be $A(x)$

Let area $ABGE$ be $A(x + h)$

Then area $DCGE$ is $A(x + h) - A(x)$

Area $DCFE <$ area $DCGE <$ area $DHGE$

$$f(x) \times h < A(x+h) - A(x) < f(x+h) \times h$$

$$f(x) < \frac{A(x+h) - A(x)}{h} < f(x+h)$$

$$\lim_{h \to 0} f(x) < \lim_{h \to 0} \frac{A(x+h) - A(x)}{h} < \lim_{h \to 0} f(x+h)$$

$$f(x) < A'(x) < f(x)$$

This is the formula for the derivative of $A(x)$ from first principles, from Year 11, Chapter 6, *Introduction to calculus*.

So $A'(x) = f(x)$

$A(x)$ is *an* anti-derivative of $f(x)$.

Let $F(x)$ be the anti-derivative of $f(x)$ with a constant term of 0.

Then $A(x) = F(x) + C$ [1]

Now $A(x)$ is the area under $y = f(x)$ between a and x.

$A(a) = 0$

Substitute in [1]:

$A(a) = F(a) + C$

$0 = F(a) + C$

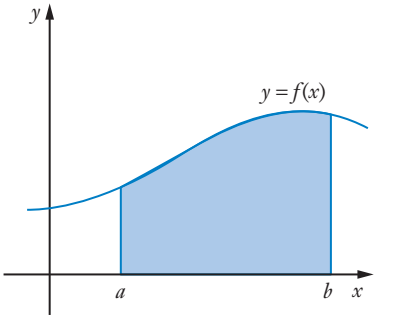
$-F(a) = C$

$A(x) = F(x) - F(a)$

If $x = b$ where $b > a$:

$A(b) = F(b) - F(a)$

So $\int_b^a f(x)\,dx = F(b) - F(a)$



EXAMPLE 8

Evaluate:

a $\int_3^4 (2x+1)\,dx$ **b** $\int_0^5 3x^2\,dx$ **c** $\int_0^2 (-3x^2)\,dx$ **d** $\int_{-1}^1 x^3\,dx$

Solution

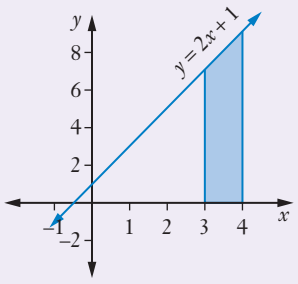
You learned in Chapter 4, *Further differentiation*, that the anti-derivative of x^n is $\frac{1}{n+1}x^{n+1} + C$. Let $F(x)$ be the anti-derivative where $C = 0$.

a $\int_3^4 (2x+1)\,dx = \left[x^2 + x\right]_3^4$ The anti-derivative

$= (4^2 + 4) - (3^2 + 3)$ $F(b) - F(a)$

$= 20 - 12$

$= 8$

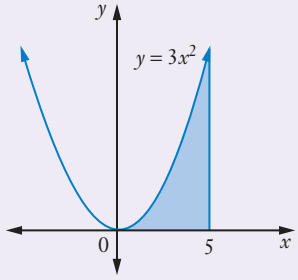


The graph shows the area that this integral calculates. Can you find its area an easier way?

b $\int_0^5 3x^2\,dx = \left[x^3\right]_0^5$

$= 5^3 - 0^3$

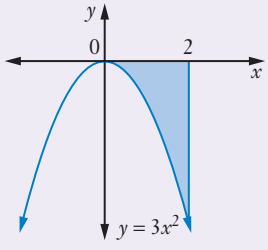
$= 125$



c $\int_0^2 (-3x^2)\,dx = \left[-x^3\right]_0^2$

$= -2^3 - (-0^3)$

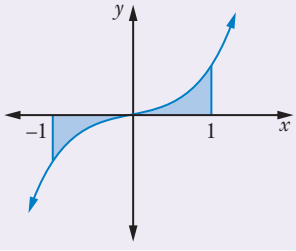
$= -8$



d $\int_{-1}^1 x^3\,dx = \left[\frac{x^4}{4}\right]_{-1}^1$

$= \frac{1^4}{4} - \frac{(-1)^4}{4}$

$= 0$



We can also find the definite integral of x^n when n is a fraction or negative.

EXAMPLE 9

Evaluate:

a $\int_1^2 \frac{2x-3}{x^3}\,dx$ **b** $\int_1^8 \sqrt[3]{x}\,dx$

ISBN 9780170413220

Solution

a
$$\int_1^2 \frac{2x-3}{x^3}\,dx = \int_1^2 \frac{2x}{x^3} - \frac{3}{x^3}\,dx$$
$$= \int_1^2 \frac{2}{x^2} - \frac{3}{x^3}\,dx$$
$$= \int_1^2 (2x^{-2} - 3x^{-3})\,dx$$
$$= \left[\frac{2x^{-1}}{-1} - \frac{3x^{-2}}{-2}\right]_1^2$$
$$= \left[-\frac{2}{x} + \frac{3}{2x^2}\right]_1^2$$
$$= \left[-\frac{2}{2} + \frac{3}{2(2)^2}\right] - \left[-\frac{2}{1} + \frac{3}{2(1)^2}\right]$$
$$= -1 + \frac{3}{8} + 2 - \frac{3}{2}$$
$$= -\frac{1}{8}$$

b
$$\int_1^8 \sqrt[3]{x}\,dx = \int_1^8 x^{\frac{1}{3}}\,dx$$
$$= \left[\frac{x^{\frac{4}{3}}}{\frac{4}{3}}\right]_1^8$$
$$= \left[\frac{3x^{\frac{4}{3}}}{4}\right]_1^8$$
$$= \left[\frac{3\sqrt[3]{x^4}}{4}\right]_1^8$$
$$= \frac{3\sqrt[3]{8^4}}{4} - \frac{3\sqrt[3]{1^4}}{4}$$
$$= \frac{3\times 16}{4} - \frac{3\times 1}{4}$$
$$= 11\frac{1}{4}$$

We can use the definite integral to find original information given a rate of change.

EXAMPLE 10

The velocity of a particle is given by $v = 8t^3 - 3t^2 + 6t + 1$ cm s^{-1}.
Find the change in displacement in the first 3 seconds.

Solution

$$v = \frac{dx}{dt} = 8t^3 - 3t^2 + 6t + 1$$
$$x = \int_0^3 (8t^3 - 3t^2 + 6t + 1)\,dt$$
$$= \left[8\frac{t^4}{4} - 3\frac{t^3}{3} + 6\frac{t^2}{2} + t\right]_0^3$$
$$= \left[2t^4 - t^3 + 3t^2 + t\right]_0^3$$
$$= [2(3)^4 - 3^3 + 3(3)^2 + 3] - [2(0)^4 - 0^3 + 3(0)^2 + 0]$$
$$= 165$$

So the change in displacement in the first 3 seconds is 165 cm.

ISBN 9780170413220

DID YOU KNOW?

Differentiation vs integration

Many mathematicians in the 17th century were interested in the problem of finding areas under a curve. The Englishman **Isaac Barrow** (1630–77) is said to be the first to discover that differentiation and integration are inverse operations. This discovery is called the **fundamental theorem of calculus**.

Barrow was an outstanding Greek scholar as well as making contributions in the areas of mathematics, theology, astronomy and physics. However, when he was a schoolboy, he was so often in trouble that his father was overheard saying to God in his prayers that if he decided to take one of his children, he could best spare Isaac.

Another English mathematician named Isaac, Sir Isaac Newton (1643–1727), was also a scientist and astronomer, and helped to discover calculus. He was not interested in his school work, but spent most of his time inventing things, such as a water clock and sundial.

Newton left school at 14 to manage the family estate after his stepfather died. However, he spent so much time reading that he was sent back to school. He went on to university and developed the theories in mathematics and science that have made him famous today.

Exercise 6.03 Definite integrals

1 Evaluate each definite integral.

a $\int_0^2 4x\,dx$ **b** $\int_1^3 (2x+1)\,dx$ **c** $\int_{-1}^6 3x^2\,dx$

d $\int_1^2 (4t-7)\,dt$ **e** $\int_{-1}^1 (6y+5)\,dy$ **f** $\int_0^3 6x^2\,dx$

g $\int_1^2 (x^2+1)\,dx$ **h** $\int_0^2 4x^3\,dx$ **i** $\int_{-1}^4 (3x^2-2x)\,dx$

2 Evaluate:

a $\int_{-1}^1 x^2\,dx$ **b** $\int_{-2}^3 (x^3+1)\,dx$ **c** $\int_{-2}^2 x^5\,dx$

d $\int_1^4 \sqrt{x}\,dx$ **e** $\int_0^1 (x^3-3x^2+4x)\,dx$ **f** $\int_1^2 (2x-1)^2\,dx$

g $\int_{-1}^1 (y^3+y)\,dy$ **h** $\int_3^4 (2-x)^2\,dx$ **i** $\int_{-2}^2 4t^3\,dt$

j $\int_2^4 \frac{x^2}{3}\,dx$ **k** $\int_1^3 \frac{5x^4}{x}\,dx$ **l** $\int_2^4 \frac{x^4-3x}{x}\,dx$

m $\int_1^2 \frac{4x^3+x^2+5x}{x}\,dx$ **n** $\int_3^5 \frac{x^3-2x^2+3x}{x}\,dx$ **o** $\int_3^4 \frac{x^2+x+3}{3x^5}\,dx$

ISBN 9780170413220

3 For each velocity function, find the change in displacement between 2 and 4 seconds.

a $v = 3t^2 + 7$ m s^{-1} **b** $v = 8t - 5$ km h^{-1}

c $v = 4t^3 + 2t + 3$ cm s^{-1} **d** $v = (t + 3)^2$ m s^{-1}

e $v = 5 - 6t + 9t^2$ cm s^{-1}

4 A high-power hose fills an empty swimming pool at the rate of $r = 25 + 4t^3$ L min^{-1}. Find the volume to the nearest litre after:

a 5 minutes **b** 15 minutes **c** half an hour

INVESTIGATION

AREAS

Look at the results of definite integrals in the examples and exercises. Sketch the graphs where possible and shade in the areas found.

Can you see why the definite integral sometimes gives a negative answer?

Can you see why it will sometimes be zero?

6.04 Indefinite integrals

To find the indefinite integral $\int f(x)\,dx$, we find the anti-derivative of the function.

WS Indefinite integrals

WS Integrating functions

Integral of x^n

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1} + C \text{ where } n \neq -1$$

EXAMPLE 11

Find each indefinite integral.

a $\int 5x^9\,dx$ **b** $\int \left(\frac{1}{x^3} + \sqrt{x}\right)dx$

Solution

a
$$\begin{aligned}\int 5x^9\,dx &= 5\int x^9\,dx \\ &= 5 \times \frac{x^{10}}{10} + C \\ &= \frac{x^{10}}{2} + C\end{aligned}$$

b
$$\begin{aligned}\int \left(\frac{1}{x^3} + \sqrt{x}\right)dx &= \int \left(x^{-3} + x^{\frac{1}{2}}\right)dx \\ &= \frac{x^{-2}}{-2} + \frac{x^{\frac{3}{2}}}{\frac{3}{2}} + C \\ &= -\frac{1}{2x^2} + \frac{2\sqrt{x^3}}{3} + C\end{aligned}$$

ISBN 9780170413220

DID YOU KNOW?

John Wallis

English clergyman and mathematician **John Wallis** (1616–1703) found that the **area under the curve** $y = 1 + x + x^2 + x^3 + \ldots$ is given by:

$$x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \ldots$$

He found this result independently of the fundamental theorem of calculus.

We can use the indefinite integral to find original information given a rate of change.

EXAMPLE 12

The rate of air flow into a container is given by

$$R = 4 + 3t^2 \text{ mm}^3 \text{ s}^{-1}$$

If there is initially no air in the container, find the volume of air in it after 12 seconds.

Solution

$R = \frac{dV}{dt} = 4 + 3t^2$

$V = \int (4 + 3t^2)\, dt$

$= 4t + t^3 + C$

When $t = 0$, $V = 0$

$0 = 4(0) + 0^3 + C$

$= C$

So $V = 4t + t^3$

When $t = 12$:

$V = 4(12) + 12^3$

$= 1776$

So the container will hold 1776 mm^3 of air after 12 seconds.

Exercise 6.04 Indefinite integrals

1 Find each indefinite integral.

a $\int x^2\, dx$ **b** $\int 3x^5\, dx$ **c** $\int 2x^4\, dx$

d $\int (m + 1)\, dm$ **e** $\int (t^2 - 7)\, dt$ **f** $\int (h^7 + 5)\, dh$

g $\int (y - 3)\, dy$ **h** $\int (2x + 4)\, dx$ **i** $\int (b^2 + b)\, db$

ISBN 9780170413220

2 Find:

a $\int (x^2 + 2x + 5)\,dx$ **b** $\int (4x^3 - 3x^2 + 8x - 1)\,dx$

c $\int (6x^5 + x^4 + 2x^3)\,dx$ **d** $\int (x^7 - 3x^6 - 9)\,dx$

e $\int (2x^3 + x^2 - x - 2)\,dx$ **f** $\int (x^5 + x^3 + 4)\,dx$

g $\int (4x^2 - 5x - 8)\,dx$ **h** $\int (3x^4 - 2x^3 + x)\,dx$

i $\int (6x^3 + 5x^2 - 4)\,dx$ **j** $\int (3x^{-4} + x^{-3} + 2x^{-2})\,dx$

3 Find each indefinite integral.

a $\int \frac{dx}{x^8}$ **b** $\int x^{\frac{1}{3}}\,dx$ **c** $\int \frac{x^6 - 3x^5 + 2x^4}{x^3}\,dx$

d $\int (1 - 2x)^2\,dx$ **e** $\int (x - 2)(x + 5)\,dx$ **f** $\int \frac{3}{x^2}\,dx$

g $\int \frac{dx}{x^3}$ **h** $\int \frac{4x^3 - x^5 - 3x^2 + 7}{x^5}\,dx$ **i** $\int (y^2 - y^{-7} + 5)\,dy$

j $\int (t^2 - 4)(t - 1)\,dt$ **k** $\int \sqrt{x}\,dx$ **l** $\int \frac{2}{t^5}\,dt$

m $\int \sqrt[3]{x}\,dx$ **n** $\int x\sqrt{x}\,dx$ **o** $\int \sqrt{x}\left(1 + \frac{1}{\sqrt{x}}\right)dx$

4 The rate of change of the angle sum S of a polygon with n sides is a constant 180°. If $S = 360°$ when $n = 4$, find S when $n = 7$.

5 For a certain graph, the rate of change of y values with respect to its x values is given by $R = 3x^2 - 2x + 1$. If the graph passes through the point (–1, 3), find its equation.

6 The rate of change in velocity over time is given by $\frac{dx}{dt} = 4t + t^2 - t^3$.

If the initial velocity is 2 cm s^{-1}, find the displacement after 15 s.

7 The rate of flow of water into a dam is given by $R = 500 + 20t$ L h^{-1}. If there is 15 000 L of water initially in the dam, how much water will there be in the dam after 10 hours?

6.05 Chain rule

You found the anti-derivative of $y = (ax + b)^n$ in Chapter 4, *Further differentiation*. We can write this as an integral.

Chain rule for $(ax + b)^n$

$$\int (ax + b)^n\,dx = \frac{(ax + b)^{n+1}}{a(n + 1)} + C \quad \text{where } n \neq -1$$

ISBN 9780170413220

EXAMPLE 13

Find:

a $\int (5x-9)^3\,dx$ **b** $\int_2^3 (3-x)^8\,dx$ **c** $\int_1^3 \sqrt{4x-3}\,dx$

Solution

a $\int (5x-9)^3\,dx = \dfrac{(5x-9)^4}{5\times 4} + C$

$= \dfrac{(5x-9)^4}{20} + C$

b $\int_2^3 (3-x)^8\,dx = \left[\dfrac{(3-x)^9}{-1\times 9}\right]_2^3$

$= \left[-\dfrac{(3-x)^9}{9}\right]_2^3$

$= -\dfrac{(3-3)^9}{9} - \left(-\dfrac{(3-2)^9}{9}\right)$

$= 0 + \dfrac{1}{9}$

$= \dfrac{1}{9}$

c $\int_1^3 \sqrt{4x-3}\,dx = \int_1^3 (4x-3)^{\frac{1}{2}}\,dx$

$= \left[\dfrac{(4x-3)^{\frac{3}{2}}}{4\times\frac{3}{2}}\right]_1^3$

$= \left[\dfrac{\sqrt{(4x-3)^3}}{6}\right]_1^3$

$= \dfrac{\sqrt{(4\times 3-3)^3}}{6} - \dfrac{\sqrt{(4\times 1-3)^3}}{6}$

$= \dfrac{\sqrt{9^3}}{6} - \dfrac{\sqrt{1^3}}{6}$

$= \dfrac{27}{6} - \dfrac{1}{6}$

$= 4\dfrac{1}{3}$

ISBN 9780170413220

In Chapter 4, *Further differentiation*, you also learned about the general chain rule for $f'(x)[f(x)]^n$.

Chain rule for $f'(x)[f(x)]^n$

$$\int f'(x)[f(x)]^n\,dx = \frac{1}{n+1}[f(x)]^{n+1} + C \quad \text{where } n \neq -1$$

EXAMPLE 14

a Find $\int x(x^2+1)^3\,dx$.

b Find the exact value of $\int_1^2 x^2\sqrt{x^3-1}\,dx$.

Solution

a $f(x) = x^2 + 1$

$f'(x) = 2x$

$$\int x(x^2+1)^3\,dx$$

$$= \int \frac{1}{2} \times 2x(x^2+1)^3\,dx$$

$$= \frac{1}{2}\int 2x(x^2+1)^3\,dx$$

$$= \frac{1}{2} \times \frac{1}{4}(x^2+1)^4 + C$$

$$= \frac{1}{8}(x^2+1)^4 + C$$

applying the chain rule formula

b $f(x) = x^3 - 1$

$f'(x) = 3x^2$

$$\int_1^2 x^2\sqrt{x^3-1}\,dx$$

$$= \int_1^2 \frac{1}{3} \times 3x^2(x^3-1)^{\frac{1}{2}}\,dx$$

$$= \frac{1}{3}\int_1^2 3x^2(x^3-1)^{\frac{1}{2}}\,dx$$

$$= \frac{1}{3}\left[\frac{1}{\frac{3}{2}}(x^3-1)^{\frac{3}{2}}\right]_1^2$$

$$= \frac{1}{3}\left[\frac{2}{3}\sqrt{(x^3-1)^3}\right]_1^2$$

$$= \frac{2}{9}\left[\sqrt{(x^3-1)^3}\right]_1^2$$

$$= \frac{2}{9}\left(\sqrt{(2^3-1)^3} - \sqrt{(1^3-1)^3}\right)$$

$$= \frac{2}{9}\left(\sqrt{7^3} - \sqrt{0^3}\right)$$

$$= \frac{2}{9}(7\sqrt{7})$$

$$= \frac{14}{9}\sqrt{7}$$

ISBN 9780170413220

Exercise 6.05 Chain rule

1 Find each indefinite integral.

a $\int (3x-4)^2\,dx$ **b** $\int (x+1)^4\,dx$ **c** $\int (5x-1)^9\,dx$

d $\int (3y-2)^7\,dy$ **e** $\int (4+3x)^4\,dx$ **f** $\int (7x+8)^{12}\,dx$

g $\int (1-x)^6\,dx$ **h** $\int \sqrt{2x-5}\,dx$ **i** $2\int (3x+1)^{-4}\,dx$

j $\int 3(x+7)^{-2}\,dx$ **k** $\int \dfrac{1}{2(4x-5)^3}\,dx$ **l** $\int \sqrt[3]{4x+3}\,dx$

m $\int (2-x)^{-\frac{1}{2}}\,dx$ **n** $\int \sqrt{(t+3)^3}\,dt$ **o** $\int \sqrt{(5x+2)^5}\,dx$

2 Evaluate:

a $\int_1^2 (2x+1)^4\,dx$ **b** $\int_0^1 (3y-2)^3\,dy$ **c** $\int_1^2 (1-x)^7\,dx$

d $\int_0^2 (3-2x)^5\,dx$ **e** $\int_0^1 \dfrac{(3x-1)^2}{6}\,dx$ **f** $\int_4^5 (5-x)^6\,dx$

g $\int_3^6 \sqrt{x-2}\,dx$ **h** $\int_0^2 \dfrac{5}{(2n+1)^3}\,dn$ **i** $\int_1^4 \dfrac{2}{\sqrt{(5x-4)^3}}\,dx$

3 Find each indefinite integral.

a $\int 4x^3(x^4+5)^2\,dx$ **b** $\int 2x(x^2-3)^5\,dx$

c $\int 3x^2(x^3+1)^3\,dx$ **d** $\int (2x+3)(x^2+3x-2)^4\,dx$

e $\int x(3x^2-7)^6\,dx$ **f** $\int x^2(4-5x^3)^2\,dx$

g $\int 4x^5(2x^6-3)^4\,dx$ **h** $\int 3x(5x^2+3)^7\,dx$

i $\int (x+2)(x^2+4x)^5\,dx$ **j** $\int (3x^2-2)(3x^3-6x-2)^3\,dy$

4 Evaluate:

a $\int_0^2 x(2x^2+3)^2\,dx$ **b** $\int_0^1 x^2(x^3-1)^5\,dx$

c $\int_1^2 x^4(x^5+2)^3\,dx$ **d** $\int_0^1 x^3(5-x^4)^7\,dx$

e $\int_2^4 3x(x^2+2)^4\,dx$ **f** $\int_{-1}^1 5x^2(2x^3-7)^3\,dx$

g $\int_{-1}^0 (x-1)(x^2-2x+3)^6\,dx$ **h** $4\int_0^1 (x^2+2)(x^3+6x-1)^2\,dx$

i $5\int_{-2}^2 x^2(x^3-1)(x^6-2x^3-1)^4\,dx$

5 A function has $\dfrac{dy}{dx} = x^2(x^3-2)^4$ and passes through the point (1, 4). Find its equation.

6 The velocity of an object is given by $v = x(x^2-3)^4$ m s^{-1}. If the displacement is 0 after 2 s, find:

a the equation for the displacement

b the displacement after 3 s.

ISBN 9780170413220

6.06 Integration involving exponential functions

We can write the anti-derivative of e^x as an integral.

Integral of e^x

$$\int e^x \, dx = e^x + C$$

EXAMPLE 15

Evaluate $\int_0^2 4e^x \, dx$.

Solution

$$\begin{aligned}\int_0^2 4e^x \, dx &= 4\left[e^x\right]_0^2 \\ &= 4(e^2 - e^0) \\ &= 4(e^2 - 1)\end{aligned}$$

Integral of a^x

$$\int a^x \, dx = \frac{1}{\ln a} a^x + C$$

EXAMPLE 16

Find $\int 2^x \, dx$.

Solution

$$\int a^x \, dx = \frac{1}{\ln a} a^x + C$$

$$\int 2^x \, dx = \frac{1}{\ln 2} 2^x + C$$

Chain rule for e^{ax+b}

$$\int e^{ax+b} \, dx = \frac{1}{a} e^{ax+b} + C$$

ISBN 9780170413220

EXAMPLE 17

a Find $\int (e^{2x} - e^{-x})\,dx$.

b Evaluate $\int_1^2 5e^{3x}\,dx$.

Solution

a $\int (e^{2x} - e^{-x})\,dx = \frac{1}{2}e^{2x} - \frac{1}{-1}e^{-1} + C$

$= \frac{1}{2}e^{2x} + e^{-x} + C$

b $\int_1^2 5e^{3x}\,dx = \left[5 \times \frac{1}{3}e^{3x}\right]_1^2$

$= \left[\frac{5e^{3x}}{3}\right]_1^2$

$= \frac{5e^{3\times 2}}{3} - \frac{5e^{3\times 1}}{3}$

$= \frac{5e^6}{3} - \frac{5e^3}{3}$

$= \frac{5e^3}{3}(e^3 - 1)$

Exercise 6.06 Integration involving exponential functions

1 Find each indefinite integral.

a $\int e^{4x}\,dx$ **b** $\int e^{-x}\,dx$ **c** $\int e^{5x}\,dx$

d $\int e^{-2x}\,dx$ **e** $\int e^{4x+1}\,dx$ **f** $\int -3e^{5x}\,dx$

g $\int e^{2t}\,dt$ **h** $\int (e^{7x} - 2)\,dx$ **i** $\int (e^{x-3} + x)\,dx$

2 Evaluate in exact form:

a $\int_0^1 e^{5x}\,dx$ **b** $\int_0^2 -e^{-x}\,dx$ **c** $\int_1^4 2e^{3x+4}\,dx$

d $\int_2^3 (3x^2 - e^{2x})\,dx$ **e** $\int_0^2 (e^{2x} + 1)\,dx$ **f** $\int_1^2 (e^x - x)\,dx$

g $\int_0^3 (e^{2x} - e^{-x})\,dx$

3 Evaluate correct to 2 decimal places:

a $\int_1^3 e^{-x}\,dx$ **b** $\int_0^2 2e^{3y}\,dy$ **c** $\int_5^6 (e^{x+5} + 2x - 3)\,dx$

d $\int_0^1 (e^{3t+4} - t)\,dt$ **e** $\int_1^2 (e^{4x} + e^{2x})\,dx$

4 Find the indefinite integral of:

a 5^x **b** 7^{3x} **c** 3^{2x-1}

ISBN 9780170413220

5 **a** Differentiate x^2e^x.

b Hence find $\int x(2+x)e^x\,dx$.

6 A function has $f'(x) = x^2e^{2x^3}$ and passes through the point (0, 0). Find the equation of the function.

7 A particle moves so that its velocity over time t is given by $v = 2e^t - 1$ m s^{-1}. If displacement $x = 10$ when $y = 0$, find x when $t = 3$.

6.07 Integration involving logarithmic functions

We can write the anti-derivatives that involve the logarithmic function as an integral.



Integrals involving exponential and logarithmic functions



Integration of $\frac{1}{x}$



Integration of reciprocals



Differentiation and integration involving exponential and logarithmic functions

Integral of $\frac{1}{x}$

$$\int \frac{1}{x}\,dx = \ln|x| + C \quad \text{where } x \neq 0$$

EXAMPLE 18

Evaluate $\int_1^5 \frac{3}{x}\,dx$.

Solution

$$\begin{aligned}\int_1^5 \frac{3}{x}\,dx &= \left[3\ln|x|\right]_1^5 \\ &= 3\ln 5 - 3\ln 1 \\ &= 3\ln 5 - 3 \times 0 \\ &= 3\ln 5\end{aligned}$$

Integral of $\frac{f'(x)}{f(x)}$

$$\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + C \quad \text{where } f(x) \neq 0$$

EXAMPLE 19

a Find $\int \frac{x^2}{x^3+7}\,dx$.

b Find the exact value of $\int_0^3 \frac{x+1}{x^2+2x+4}\,dx$.

ISBN 9780170413220

Solution

a $f(x) = x^3 + 7,\ f'(x) = 3x^2$

$$\int \frac{x^2}{x^3+7}\,dx = \int \frac{1}{3} \times \frac{3x^2}{x^3+7}\,dx$$

$$= \frac{1}{3}\int \frac{3x^2}{x^3+7}\,dx$$

$$= \frac{1}{3}\ln|x^3+7| + C$$

← applying the formula

b $f(x) = x^2 + 2x + 4,\ f'(x) = 2x + 2$

$$\int_0^3 \frac{x+1}{x^2+2x+4}\,dx = \frac{1}{2}\int_0^3 \frac{2x+2}{x^2+2x+4}\,dx \qquad \text{as } x+1 = \frac{1}{2}(2x+2)$$

$$= \frac{1}{2}\Big[\ln|x^2+2x+4|\Big]_0^3$$

$$= \frac{1}{2}[\ln|3^2+2(3)+4| - \ln|0^2+2(0)+4|]$$

$$= \frac{1}{2}[\ln 19 - \ln 4]$$

$$= \frac{1}{2}\ln\left(\frac{19}{4}\right)$$

Exercise 6.07 Integration involving logarithmic functions

1 Find the integral of each function.

a $\dfrac{2}{2x+5}$ **b** $\dfrac{4x}{2x^2+1}$ **c** $\dfrac{5x^4}{x^5-2}$ **d** $\dfrac{1}{2x}$ **e** $\dfrac{2}{x}$

f $\dfrac{5}{3x}$ **g** $\dfrac{2x-3}{x^2-3x}$ **h** $\dfrac{x}{x^2+2}$ **i** $\dfrac{3x}{x^2+7}$ **j** $\dfrac{x+1}{x^2+2x-5}$

2 Find:

a $\displaystyle\int \frac{4}{4x-1}\,dx$ **b** $\displaystyle\int \frac{dx}{x+3}$ **c** $\displaystyle\int \frac{x^2}{2x^3-7}\,dx$

d $\displaystyle\int \frac{x^5}{2x^6+5}\,dx$ **e** $\displaystyle\int \frac{x+3}{x^2+6x+2}\,dx$

3 Evaluate correct to one decimal place:

a $\displaystyle\int_1^3 \frac{2}{2x+5}\,dx$ **b** $\displaystyle\int_2^5 \frac{dx}{x+1}$ **c** $\displaystyle\int_1^7 \frac{x^2}{x^3+2}\,dx$

d $\displaystyle\int_0^3 \frac{4x+1}{2x^2+x+1}\,dx$ **e** $\displaystyle\int_3^4 \frac{x-1}{x^2-2x}\,dx$

ISBN 9780170413220

4 a Show that $\frac{3x+3}{x^2-9} = \frac{1}{x+3} + \frac{2}{x-3}$.

b Hence find $\int \frac{3x+3}{x^2-9}\,dx$.

5 a Show that $\frac{x-6}{x-1} = 1 - \frac{5}{x-1}$.

b Hence find $\int \frac{x-6}{x-1}\,dx$.

6 A function has $f'(x) = \frac{x^2}{3x^3-1}$ and passes through the point (1, 0). Find the equation of the function.

7 A particle has velocity $v = \frac{5t}{t^2+4}$ m s^{-1}. Find its displacement after 5 s if its initial displacement is 4 m.

8 The number of people with measles is increasing at the rate given by $R = \frac{x^2}{3x^3+1}$ people/week. If 3 people had measles initially, find the number with measles after 8 weeks.

6.08 Integration involving trigonometric functions

Integrals of trigonometric functions

$$\int \cos x\,dx = \sin x + C$$

$$\int \sin x\,dx = -\cos x + C$$

$$\int \sec^2 x\,dx = \tan x + C$$

WS Integrals of trigonometric functions

WS Mixed integration questions

WS Finding indefinite integrals 1

WS Finding indefinite integrals 2

WS Finding definite integrals

EXAMPLE 20

Find the exact value of $\int_0^{\frac{\pi}{3}} \sec^2 x\,dx$.

Solution

$$\int_0^{\frac{\pi}{3}} \sec^2 x\,dx = [\tan x]_0^{\frac{\pi}{3}}$$

$$= \tan\frac{\pi}{3} - \tan 0$$

$$= \sqrt{3}$$

ISBN 9780170413220

Chain rule for trigonometric functions

$$\int \cos(ax+b)\,dx = \frac{1}{a}\sin(ax+b) + C$$

$$\int \sin(ax+b)\,dx = -\frac{1}{a}\cos(ax+b) + C$$

$$\int \sec^2(ax+b)\,dx = \frac{1}{a}\tan(ax+b) + C$$

EXAMPLE 21

a Find $\int \sin 3x\,dx$

b Find $\int \cos x^\circ\,dx$.

c Find the exact value of $\int_0^{\frac{\pi}{8}} \sin 2x\,dx$.

Solution

a $\int \sin 3x\,dx = -\frac{1}{3}\cos 3x + C$

b $\int \cos x^\circ\,dx = \int \cos\left(\frac{\pi x}{180}\right)dx$

$$= \frac{1}{\frac{\pi}{180}}\sin\left(\frac{\pi x}{180}\right) + C$$

$$= \frac{180}{\pi}\sin x^\circ + C$$

c $\int_0^{\frac{\pi}{8}} \sin 2x\,dx = \left[-\frac{1}{2}\cos 2x\right]_0^{\frac{\pi}{8}}$

$$= -\frac{1}{2}\cos\left(2\times\frac{\pi}{8}\right) - \left[-\frac{1}{2}\cos(2\times 0)\right]$$

$$= -\frac{1}{2}\cos\frac{\pi}{4} + \frac{1}{2}\cos 0$$

$$= -\frac{1}{2}\times\frac{1}{\sqrt{2}} + \frac{1}{2}$$

$$= -\frac{1}{2\sqrt{2}} + \frac{1}{2}$$

$$= \frac{2-\sqrt{2}}{4}$$

Exercise 6.08 Integration involving trigonometric functions

1 Find the integral of each function.

a	$\cos x$	**b**	$\sin x$	**c**	$\sec^2 x$
d	$\frac{\sin x^\circ}{4}$	**e**	$\sin 3x$	**f**	$-\sin 7x$
g	$\sec^2 5x$	**h**	$\cos(x+1)$	**i**	$\sin(2x-3)$
j	$\cos(2x-1)$	**k**	$\sin(\pi - x)$	**l**	$\cos(x+\pi)$
m	$2\sec^2 7x$	**n**	$4\sin\left(\frac{x}{2}\right)$	**o**	$3\sec^2\left(\frac{x}{3}\right)$

ISBN 9780170413220

2 Evaluate each definite integral, giving exact answers where appropriate.

a $\int_0^{\frac{\pi}{2}} \cos x \, dx$ **b** $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \sec^2 x \, dx$ **c** $\int_{\frac{\pi}{2}}^{\pi} \sin \frac{x}{2} \, dx$

d $\int_0^{\frac{\pi}{2}} \cos 3x \, dx$ **e** $\int_0^{\frac{1}{2}} \sin(\pi x) \, dx$ **f** $\int_0^{\frac{\pi}{8}} \sec^2 2x \, dx$

g $\int_0^{\frac{\pi}{12}} 3 \cos 2x \, dx$ **h** $\int_0^{\frac{\pi}{10}} -\sin(5x) \, dx$

3 Find:

a $\int \left(\cos x \cos \frac{\pi}{3} - \sin x \sin \frac{\pi}{3} \right) dx$ **b** $\int (\sin \pi \cos x - \cos \pi \sin x) \, dx$

4 A curve has $\frac{dy}{dx} = \cos 4x$ and passes through the point $(\pi, \frac{\pi}{4})$.

Find the equation of the curve.

5 A pendulum swings at the rate given by $\frac{dx}{dt} = 12\pi \cos \frac{2\pi t}{3}$ cm s^{-1}.

It starts 2 cm to the right of the origin.

a Find the equation of the displacement of the pendulum.

b Find the exact displacement after:

i 1 s **ii** 5 s

6 The rate at which the depth of water changes in a bay is given by $R = 4\pi \sin \frac{\pi t}{6}$ m h^{-1}.

a Find the equation of the depth of water d over time t hours if the depth is 2 m initially.

b Find the depth after 2 hours.

c Find the highest, lowest and centre of depth of water.

d What is the period of the depth of water?



iStock.com/CBCK-Christine

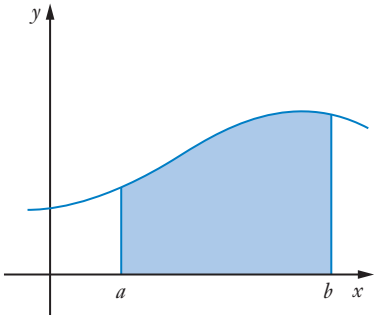
Starting areas

Area under a curve

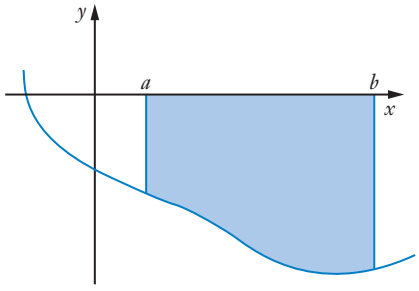
6.09 Areas enclosed by the *x*-axis

The definite integral gives the **signed** area under a curve.

Areas above the *x*-axis give a positive definite integral.



Areas below the *x*-axis give a negative definite integral.



So, to find areas below the *x*-axis, we take the absolute value of the definite integral.

Area under a curve

$$\text{Area} = \left| \int_a^b f(x)\,dx \right|$$

It is important to sketch the graph to see where the area is in relation to the *x*-axis.



Shutterstock.com/Yuriy Golub

ISBN 9780170413220

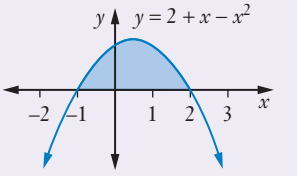
EXAMPLE 22

a Find the area enclosed by the curve $y = 2 + x - x^2$ and the x-axis.

b Find the area bounded by the curve $y = x^2 - 4$ and the x-axis.

Solution

a Sketch the graph of $y = 2 + x - x^2$ and shade the area enclosed between the curve and the x-axis.



The area is above the x-axis, so the definite integral will be positive.

$$\text{Area} = \int_{-1}^{2} 2 + x - x^2 \, dx$$

$$= \left[2x + \frac{x^2}{2} - \frac{x^3}{3} \right]_{-1}^{2}$$

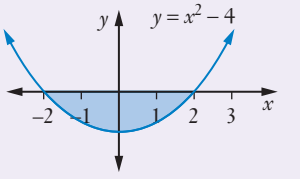
$$= \left(2(2) + \frac{2^2}{2} - \frac{2^3}{3} \right) - \left(2(-1) + \frac{(-1)^2}{2} - \frac{(-1)^3}{3} \right)$$

$$= \left(4 + 2 - \frac{8}{3} \right) - \left(-2 + \frac{1}{2} + \frac{1}{3} \right)$$

$$= 4\frac{1}{2}$$

So the area is $4\frac{1}{2}$ units^2.

b Sketch the graph of $y = x^2 - 4$.



The definite integral will be negative because the area is below the x-axis.

$$\int_{-2}^{2} (x^2 - 4) \, dx = \left[\frac{x^3}{3} - 4x \right]_{-2}^{2}$$

$$= \left(\frac{2^3}{3} - 4(2) \right) - \left(\frac{(-2)^3}{3} - 4(-2) \right)$$

$$= \left(\frac{8}{3} - 8 \right) - \left(-\frac{8}{3} + 8 \right)$$

$$= -10\frac{2}{3}$$

$$\text{Area} = \left| -10\frac{2}{3} \right|$$

$$= 10\frac{2}{3} \text{ units}^2$$

ISBN 9780170413220

EXAMPLE 23

Area under a curve

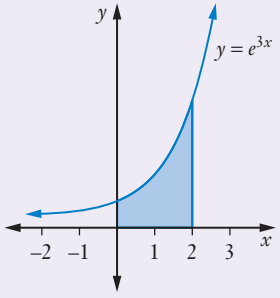
a Find the exact area enclosed between the curve $y = e^{3x}$, the x-axis and the lines $x = 0$ and $x = 2$.

b Find the exact area enclosed between the hyperbola $y = \frac{1}{x}$, the x-axis and the lines $x = 1$ and $x = 2$.

c Find the area enclosed between the curve $y = \cos x$, the x-axis and the lines $x = \frac{\pi}{2}$ and $x = \frac{3\pi}{2}$.

Solution

a Sketch the graph of $y = e^{3x}$.

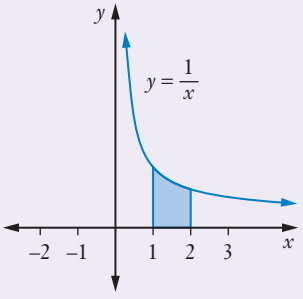


The definite integral will be positive because the area is above the x-axis.

$$\text{Area} = \int_0^2 e^{3x}\,dx$$
$$= \left[\frac{1}{3}e^{3x}\right]_0^2$$
$$= \frac{1}{3}e^{3\times 2} - \frac{1}{3}e^{3\times 0}$$
$$= \frac{1}{3}e^{6} - \frac{1}{3}e^{0}$$
$$= \frac{1}{3}(e^6 - 1)$$

So the area is $\frac{1}{3}(e^6 - 1)$ units^2.

b Sketch the graph of $y = \frac{1}{x}$. For $x = 1$ and $x = 2$, we only need to sketch the graph in the 1st quadrant.



The definite integral will be positive because the area is above the x-axis.

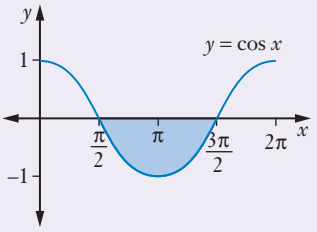
$$\text{Area} = \int_1^2 \frac{dx}{x}$$
$$= \left[\ln|x|\right]_1^2$$
$$= \ln 2 - \ln 1$$
$$= \ln 2 - 0$$
$$= \ln 2$$

Absolute value not required as 2 and 1 are positive.

So the area is $\ln 2$ units^2.

ISBN 9780170413220

c Sketch the graph of $y = \cos x$.



The definite integral will be negative because the area is below the x-axis.

$$\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \cos x \, dx = [\sin x]_{\frac{\pi}{2}}^{\frac{3\pi}{2}}$$

$$= \sin\left(\frac{3\pi}{2}\right) - \sin\left(\frac{\pi}{2}\right)$$

$$= -1 - 1$$

$$= -2$$

$$\text{Area} = |-2|$$

$$= 2 \text{ units}^2$$

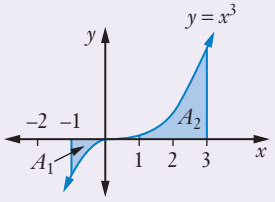
If the area has some parts above the x-axis and some below the x-axis, we need to find these separately.

EXAMPLE 24

Find the area enclosed between the curve $y = x^3$, the x-axis and the lines $x = -1$ and $x = 3$.

Solution

Sketch the graph of $y = x^3$.



There are 2 areas, marked A_1 and A_2 on the diagram.

A_1 is below the x-axis so the integral will be negative.

A_2 is above the x-axis so the integral will be positive.

A_1:

$$\int_{-1}^{0} x^3 \, dx = \left[\frac{x^4}{4}\right]_{-1}^{0}$$

$$= \frac{0^4}{4} - \frac{(-1)^4}{4}$$

$$= -\frac{1}{4}$$

$$\text{So } A_1 = \left|-\frac{1}{4}\right|$$

$$= \frac{1}{4} \text{ units}^2$$

$$A_2 = \int_0^3 x^3 \, dx$$

$$= \left[\frac{x^4}{4}\right]_0^3$$

$$= \frac{3^4}{4} - \frac{0^4}{4}$$

$$= \frac{81}{4} \text{ units}^2$$

$$\text{Total area} = A_1 + A_2$$

$$= \frac{1}{4} + \frac{81}{4}$$

$$= 20\frac{1}{2} \text{ units}^2$$

ISBN 9780170413220

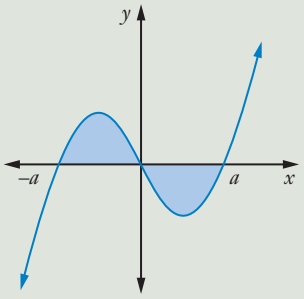
Odd and even functions

Some functions have special properties that we can use to find their areas.

Odd functions

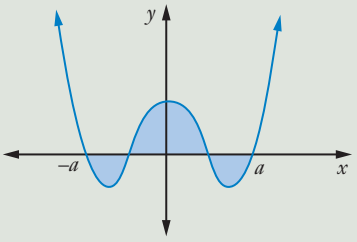
If $f(-x) = -f(x)$, then $\int_{-a}^{a} f(x)\,dx = 0$.

The positive and negative areas cancel each other out.



Even functions

If $f(-x) = f(x)$, then $\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$.



EXAMPLE 25

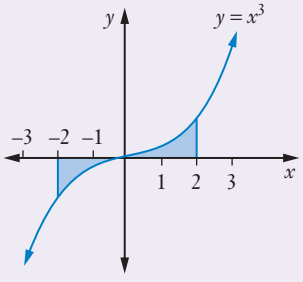
Find the area between the curve:

a $y = x^3$, the x-axis and the lines $x = -2$ and $x = 2$

b $y = x^2$, the x-axis and the lines $x = -4$ and $x = 4$

Solution

a Sketch the graph of $y = x^3$ and shade the area bounded by the curve, the x-axis and the boundaries $x = \pm 2$.



$y = x^3$ is an odd function since $f(-x) = -f(x)$.

This means that the shaded areas are symmetrical.
We can find the area between $x = 0$ and $x = 2$.

The total area will be twice this area.

ISBN 9780170413220

$$\text{Area} = 2\int_0^2 x^3\,dx$$

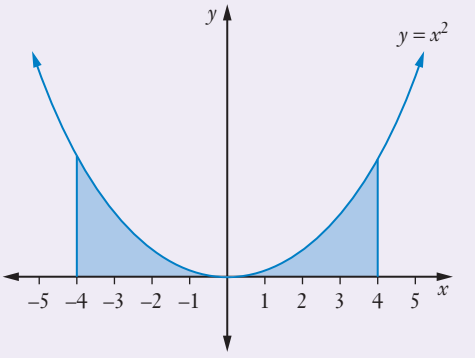
$$= 2\left[\frac{x^4}{4}\right]_0^2$$

$$= 2\left(\frac{2^4}{4} - \frac{0^4}{4}\right)$$

$$= 8$$

So area is 8 units^2.

b Sketch the graph of $y = x^2$ and shade the area enclosed between the curve, the x-axis and the lines $x = \pm 4$.



$y = x^2$ is an even function since $f(-x) = f(x)$.

$$\text{Area} = \int_{-4}^4 x^2\,dx$$

$$= 2\int_0^4 x^2\,dx$$

$$= 2\left[\frac{x^3}{3}\right]_0^4$$

$$= 2\left(\frac{4^3}{3} - \frac{0^3}{3}\right)$$

$$= 2 \times \frac{64}{3}$$

$$= 42\frac{2}{3}$$

So area is $42\frac{2}{3}$ units^2.

Exercise 6.09 Areas enclosed by the *x*-axis

1 Find the area enclosed between the curve $y = 1 - x^2$ and the x-axis.

2 Find the area bounded by the curve $y = x^2 - 9$ and the x-axis.

3 Find the area enclosed between the curve $y = x^2 + 5x + 4$ and the x-axis.

4 Find the area enclosed between the curve $y = x^2 - 2x - 3$ and the x-axis.

5 Find the area bounded by the curve $y = -x^2 + 9x - 20$ and the x-axis.

6 Find the area enclosed between the curve $y = -2x^2 - 5x + 3$ and the x-axis.

7 Find the area enclosed between the curve $y = x^3$, the x-axis and the lines $x = 0$ and $x = 2$.

ISBN 9780170413220

8 Find the area enclosed between the curve $y = x^4$, the x-axis and the lines $x = -1$ and $x = 1$.

9 Find the area enclosed between the curve $y = x^3$, the x-axis and the lines $x = -2$ and $x = 2$.

10 Find the area enclosed between the curve $y = x^3$, the x-axis and the lines $x = -3$ and $x = 2$.

11 Find the exact area enclosed by the curve $y = 2e^{2x}$, the x-axis and the lines $x = 1$ and $x = 2$.

12 Find the exact area bounded by the curve $y = e^{4x-3}$, the x-axis and the lines $x = 0$ and $x = 1$.

13 Find the area enclosed by the curve $y = x + e^{-x}$, the x-axis and the lines $x = 0$ and $x = 2$, correct to 2 decimal places.

14 Find the area bounded by the curve $y = e^{5x}$, the x-axis and the lines $x = 0$ and $x = 1$, correct to 3 significant figures.

15 Find the area enclosed between the curve $y = \sin x$ and the x-axis in the domain $[0, 2\pi]$.

16 Find the exact area bounded by the curve $y = \cos 3x$, the x-axis and the lines $x = 0$ and $x = \dfrac{\pi}{12}$.

17 Find the area enclosed between the curve $y = \sec^2 \dfrac{x}{4}$, the x-axis and the lines $x = \dfrac{\pi}{4}$ and $x = \dfrac{\pi}{2}$, correct to 2 decimal places.

18 Find the area bounded by the curve $y = 3x^2$, the x-axis and the lines $x = -1$ and $x = 1$.

19 Find the area enclosed between the curve $y = x^2 + 1$, the x-axis and the lines $x = -2$ and $x = 2$.

20 Find the area enclosed between the curve $y = x^2$, the x-axis and the lines $x = -3$ and $x = 2$.

21 Find the area enclosed between the curve $y = x^2 + x$, and the x-axis.

22 Find the area enclosed between the curve $y = \dfrac{1}{x^2}$, the x-axis and the lines $x = 1$ and $x = 3$.

23 Find the area enclosed between the curve $y = \dfrac{2}{(x-3)^2}$, the x-axis and the lines $x = 0$ and $x = 1$.

24 Find the exact area between the curve $y = \dfrac{1}{x}$, the x-axis and the lines $x = 2$ and $x = 3$.

25 Find the exact area bounded by the curve $y = \dfrac{1}{x-1}$, the x-axis and the lines $x = 4$ and $x = 7$.

26 Find the area bounded by the curve $y = \dfrac{x}{x^2+1}$, the x-axis and the lines $x = 2$ and $x = 4$, correct to 2 decimal places.

27 Find the area bounded by the curve $y = \sqrt{x}$, the x-axis and the line $x = 4$.

28 Find the area bounded by the curve $y = \sqrt{x+2}$, the x-axis and the line $x = 7$.

29 Use the trapezoidal rule with 4 subintervals to find the area bounded by the curve $y = \ln x$, the x-axis and the line $x = 5$, correct to 2 decimal places.

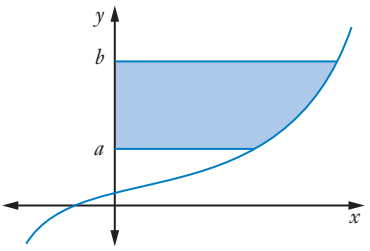
30 Find the area bounded by the x-axis, the curve $y = x^3$ and the lines $x = -a$ and $x = a$.

ISBN 9780170413220

6.10 Areas enclosed by the *y*-axis

WS
Areas of integration

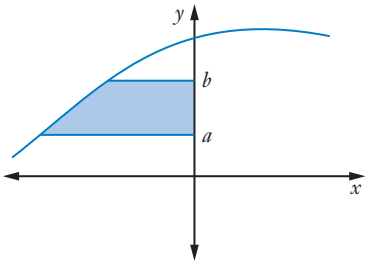
We can find an area bounded by a graph and the *y*-axis by writing the equation in the form $x = f(y)$.



The definite integral gives the **signed** area.

Areas to the right of the *y*-axis give a positive definite integral.

Areas to the left of the *y*-axis give a negative definite integral.



Area bounded by a curve and the *y*-axis

For the curve $x = f(y)$:

$$\text{Area} = \left| \int_a^b f(y)\,dy \right|$$

EXAMPLE 26

a Find the area enclosed by the curve $x = y^2$, the *y*-axis and the lines $y = 1$ and $y = 3$.

b Find the area enclosed by the curve $y = x^2$, the *y*-axis and the lines $y = 0$ and $y = 4$ in the first quadrant.

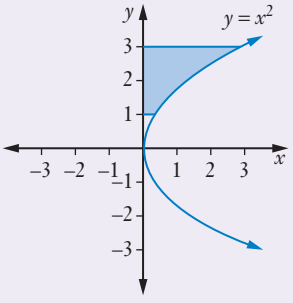
Solution

a Sketch the graph of $x = y^2$ and shade the area bounded by the curve, the *y*-axis and the lines $y = 1$ and $y = 3$.

This is the same shape as the parabola $y = x^2$ with the x and y values swapped.

For example, when $x = 1$, $y = \pm 1$, when $x = 4$, $y = \pm 2$.

The area is to the right of the *y*-axis so the integral will be positive.

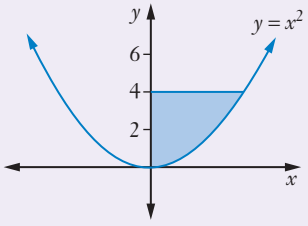


ISBN 9780170413220

$$\text{Area} = \int_a^b f(y)\,dy$$
$$= \int_1^3 y^2\,dy$$
$$= \left[\frac{y^3}{3}\right]_1^3$$
$$= \frac{3^3}{3} - \frac{1^3}{3}$$
$$= 8\frac{2}{3}$$

So the area is $8\frac{2}{3}$ units^2

b Sketch the graph of $y = x^2$ and shade the area enclosed between the curve, the y-axis and the lines $y = 0$ and $y = 4$.



The area is to the right of the y-axis so the integral will be positive.

Change the subject of the equation to x.

$$y = x^2$$
$$\pm\sqrt{y} = x$$

In the first quadrant:

$$x = \sqrt{y}$$
$$= y^{\frac{1}{2}}$$
$$\text{Area} = \int_a^b f(y)\,dy$$
$$= \int_0^4 y^{\frac{1}{2}}\,dy$$
$$= \left[\frac{y^{\frac{3}{2}}}{\frac{3}{2}}\right]_0^4$$
$$= \left[\frac{2\sqrt{y^3}}{3}\right]_0^4$$
$$= \frac{2\sqrt{4^3}}{3} - \frac{2\sqrt{0^3}}{3}$$
$$= 5\frac{1}{3}$$

So the area is $5\frac{1}{3}$ units^2.

ISBN 9780170413220

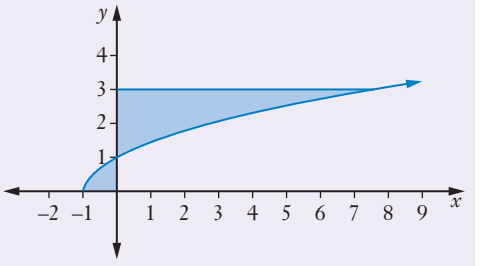
EXAMPLE 27

Find the area enclosed between the curve $y = \sqrt{x+1}$, the y-axis and the lines $y = 0$ and $y = 3$.

Solution

The area between $y = 0$ and $y = 1$ is to the left of the y-axis so the integral will be negative.

The area between $y = 1$ and $y = 3$ is to the right of the y-axis so the integral will be positive.



Changing the subject:

$$y = \sqrt{x+1}$$
$$y^2 = x + 1$$
$$y^2 - 1 = x$$

$$\int_0^1 (y^2 - 1)\,dy = \left[\frac{y^3}{3} - y\right]_0^1$$
$$= \left(\frac{1^3}{3} - 1\right) - \left(\frac{0^3}{3} - 0\right)$$
$$= -\frac{2}{3}$$

$$A_1 = \left|-\frac{2}{3}\right| = \frac{2}{3} \text{ units}^2$$

$$A_2 = \int_1^3 (y^2 - 1)\,dy = \left[\frac{y^3}{3} - y\right]_1^3$$
$$= \left(\frac{3^3}{3} - 3\right) - \left(\frac{1^3}{3} - 1\right)$$
$$= 6\frac{2}{3} \text{ units}^2$$

$$\text{Total area} = A_1 + A_2 = \frac{2}{3} + 6\frac{2}{3} = 7\frac{1}{3} \text{ units}^2$$

Exercise 6.10 Areas enclosed by the y-axis

1 Find the area bounded by the y-axis, the curve $x = y^2$ and the lines $y = 0$ and $y = 4$.

2 Find the area enclosed between the curve $x = y^3$, the y-axis and the lines $y = 1$ and $y = 3$.

3 Find the area in the first quadrant enclosed between the curve $y = x^2$, the y-axis and the lines $y = 1$ and $y = 4$.

4 Find the area between the lines $y = x - 1$, $y = 0$, $y = 1$ and the y-axis.

ISBN 9780170413220

5 Find the area bounded by the line $y = 2x + 1$, the y-axis and the lines $y = 3$ and $y = 4$.

6 Find the area bounded by the curve $y = \sqrt{x}$, the y-axis and the lines $y = 1$ and $y = 2$.

7 Find the area bounded by the curve $x = y^2 - 2y - 3$ and the y-axis.

8 Find the area bounded by the curve $x = -y^2 - 5y - 6$ and the y-axis.

9 Find the area enclosed by the curve $y = \sqrt{3x - 5}$, the y-axis and the lines $y = 2$ and $y = 3$.

10 Find the area in the first quadrant enclosed between the curve $y = \dfrac{1}{x^2}$, the y-axis and the lines $y = 1$ and $y = 4$.

11 Find the area enclosed between the curve $y = x^3$, the y-axis and the lines $y = 1$ and $y = 8$.

12 Find the area enclosed between the curve $y = x^3 - 2$ and the y-axis between $y = -1$ and $y = 25$.

13 Find the area in the second quadrant enclosed between the lines $y = 4$ and $y = 1 - x$.

14 Find the area enclosed between the y-axis and the curve $x = y(y - 2)$.

15 Find the area in the first quadrant bounded by the curve $y = x^4 + 1$, the y-axis and the lines $y = 1$ and $y = 3$, correct to 2 significant figures.

16 Find the area between the curve $y = \ln x$, the y-axis and the lines $y = 2$ and $y = 4$, correct to 3 significant figures.

WS Sums and differences of areas

WS Calculating physical areas

WS Calculating areas between curves

WS Areas between curves

WS Areas between curves 2

6.11 Sums and differences of areas

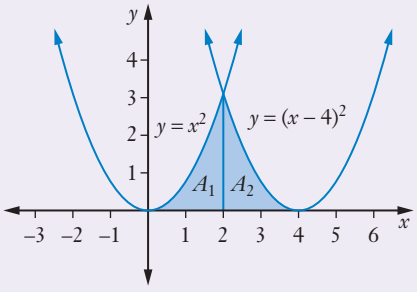
EXAMPLE 28

a Find the area enclosed between the curves $y = x^2$, $y = (x - 4)^2$ and the x-axis.

b Find the area enclosed between the curve $y = x^2$ and the line $y = x + 2$.

Solution

a Sketch the graphs of $y = x^2$ and $y = (x - 4)^2$ (translation of $y = x^2$ by 4 units to the right) and shade the area enclosed between the curves and the x-axis.



Find the x values of their intersection ($x = 2$, from the graph or by solving simultaneous equations).

Shaded area $= A_1 + A_2$

$$= \int_0^2 x^2\,dx + \int_2^4 (x - 4)^2\,dx$$

ISBN 9780170413220

Area A_1:

$$\int_0^2 x^2\,dx = \left[\frac{x^3}{3}\right]_0^2$$

$$= \frac{2^3}{3} - \frac{0^3}{3}$$

$$= 2\frac{2}{3}$$

Area A_2:

$$\int_2^4 (x-4)^2\,dx = \left[\frac{(x-4)^3}{3}\right]_2^4$$

$$= \frac{(4-4)^3}{3} - \frac{(2-4)^3}{3}$$

$$= 0 + \frac{8}{3}$$

$$= 2\frac{2}{3}$$

Total area $= A_1 + A_2$

$$= 2\frac{2}{3} + 2\frac{2}{3}$$

$$= 5\frac{1}{3}$$

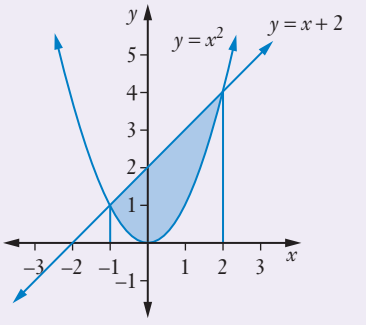
So area is $5\frac{1}{3}$ units^2.

b Sketch the graphs of $y = x^2$ and $x + 2$ and shade the area enclosed between them.

We can find the x values of the points of intersection of the functions from the graph or by solving simultaneous equations:

$y = x^2$ [1]

$y = x + 2$ [2]



Substituting [1] into [2]:

$$x^2 = x + 2$$

$$x^2 - x - 2 = 0$$

$$(x-2)(x+1) = 0$$

$$x = 2, \qquad x = -1$$

Notice that between $x = -1$ and $x = 2$, the graph of $y = x + 2$ is *above* the graph of $y = x^2$.

So we can find the area by integrating $(x + 2)$ and x^2 between $x = -1$ and $x = 2$ and then finding their difference.

ISBN 9780170413220

$$A = \int_{-1}^{2} (x+2)\,dx - \int_{-1}^{2} x^2\,dx = \int_{-1}^{2} (x+2-x^2)\,dx$$

$$= \left[\frac{x^2}{2} + 2x - \frac{x^3}{3}\right]_{-1}^{2}$$

$$= \left[\frac{2^2}{2} + 2(2) - \frac{2^3}{3}\right] - \left[\frac{(-1)^2}{2} + 2(-1) - \frac{(-1^3)}{3}\right]$$

$$= \frac{10}{3} - \left(\frac{-7}{6}\right)$$

$$= \frac{9}{2}$$

$$= 4\frac{1}{2}$$

So area is $4\frac{1}{2}$ units^2.

Exercise 6.11 Sums and differences of areas

1 Find the area bounded by the line $y = 1$ and the curve $y = x^2$.

2 Find the area enclosed between the line $y = 2$ and the curve $y = x^2 + 1$.

3 Find the area enclosed by the curve $y = x^2$ and the line $y = x$.

4 Find the area bounded by the curve $y = 9 - x^2$ and the line $y = 5$.

5 Find the area enclosed between the curve $y = x^2$ and the line $y = x + 6$.

6 Find the area bounded by the curve $y = x^3$ and the line $y = 4x$.

7 Find the area enclosed between the curves $y = (x - 1)^2$ and $y = (x + 1)^2$ and the x-axis.

8 Find the area enclosed between the curve $y = x^2$ and the line $y = -6x + 16$.

9 Find the area enclosed between the curve $y = x^3$, the x-axis and the line $y = -3x + 4$.

10 Find the area enclosed by the curves $y = (x - 2)^2$ and $y = (x - 4)^2$.

ISBN 9780170413220

11 Find the area enclosed between the curves $y = x^2$ and $y = x^3$.

12 Find the area enclosed by the curves $y = x^2$ and $x = y^2$.

13 Find the area bounded by the curve $y = x^2 + 2x - 8$ and the line $y = 2x + 1$.

14 Find the area bounded by the curves $y = 1 - x^2$ and $y = x^2 - 1$.

15 Find the exact area enclosed between the curve $y = \sqrt{4 - x^2}$ and the line $x - y + 2 = 0$.

16 Find the exact area in the first quadrant between the curve $y = \frac{1}{x}$, the x-axis and the lines $y = x$ and $x = 2$.

17 Find the exact area bounded by the curves $y = \sin x$ and $y = \cos x$ in the domain $[0, 2\pi]$.

18 Find the exact area enclosed between the curve $y = e^{2x}$ and the lines $y = 1$ and $x = 2$.

19 Find the exact area enclosed by the curve $y = \sin x$ and the line $y = \frac{1}{2}$ for $[0, 2\pi]$.

Summary of integration rules

Rule	Chain rule
$\int x^n \, dx = \frac{1}{n+1} x^{n+1} + C$	$\int f'(x)[f(x)]^n \, dx = \frac{1}{n+1}\left[f(x)^{n+1}\right] + C$
$\int e^x \, dx = e^x + C$	$\int e^{ax+b} \, dx = \frac{1}{a} e^{ax+b} + C$
$\int a^x \, dx = \frac{1}{\ln a} a^x + C$	
$\int \frac{1}{x} dx = \ln\lvert x\rvert + C$	$\int \frac{f'(x)}{f(x)} dx = \ln\lvert f(x)\rvert + C$
$\int \cos x \, dx = \sin x + C$	$\int \cos(ax+b) \, dx = \frac{1}{a} \sin(ax+b) + C$
$\int \sin x \, dx = -\cos x + C$	$\int \sin(ax+b) \, dx = -\frac{1}{a} \cos(ax+b) + C$
$\int \sec^2 x \, dx = \tan x + C$	$\int \sec^2(ax+b) \, dx = \frac{1}{a} \tan(ax+b) + C$

ISBN 9780170413220

6. TEST YOURSELF

WS Derivatives and Integrals

Qz Practice quiz

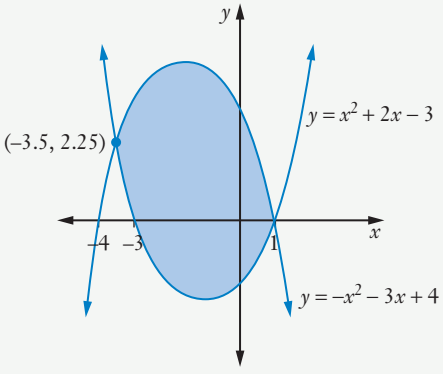
For Questions **1** to **4**, choose the correct answer **A**, **B**, **C** or **D**.

1 Find $\int \sin(6x)\,dx$.

A $\frac{1}{6}\cos(6x) + C$ **B** $6\cos(6x) + C$

C $-6\cos(6x) + C$ **D** $-\frac{1}{6}\cos(6x) + C$

2 Find the shaded area below.



A $\int_{-4}^{1}(-x^2-3x+4)\,dx - \int_{-4}^{1}(x^2+2x-3)\,dx$

B $\int_{-3.5}^{1}(-x^2-3x+4)\,dx - \int_{-3.5}^{1}(x^2+2x-3)\,dx$

C $\int_{-3}^{1}(-x^2-3x+4)\,dx + \int_{-3}^{1}(x^2+2x-3)\,dx$

D $\int_{-3.5}^{1}(-x^2-3x+4)\,dx + \int_{-3.5}^{1}(x^2+2x-3)\,dx$

3 Find $\int 4e^{3x}\,dx$.

A $\frac{4}{3}e^{3x} + C$ **B** $\frac{3}{4}e^{3x} + C$ **C** $12e^{3x} + C$ **D** $\frac{1}{12}e^{3x} + C$

4 Find $\int \frac{x}{x^2+3}dx$.

A $\frac{2}{(x^2+3)^2} + C$ **B** $2\ln|x^2+3| + C$

C $\frac{1}{2(x^2+3)^2}$ **D** $\frac{1}{2}\ln|x^2+3| + C$

5 **a** Use the trapezoidal rule with 2 subintervals to find an approximation to $\int_1^2 \frac{dx}{x^2}$.

b Use integration to find the exact value of $\int_1^2 \frac{dx}{x^2}$.

6 Find the integral of:

a $3x + 1$ **b** $\frac{5x^2 - x}{x}$ **c** $\sqrt{x}$

d $(2x+5)^7$ **e** $x^3(3x^4-2)^4$

ISBN 9780170413220

7 Find $\int 3^x \, dx$.

8 Find the approximate area under the curve $f(x) = x^3$ between $x = 1$ and $x = 3$ by using:

a 4 inner rectangles **b** 4 outer rectangles **c** a trapezium

9 Evaluate:

a $\int_0^2 (x^3 - 1) \, dx$ **b** $\int_{-1}^1 x^5 \, dx$ **c** $\int_0^1 (3x - 1)^4 \, dx$

d $\int_0^1 x^2 (x^3 - 5)^2 \, dx$ **e** $\int_{-1}^2 3x(x^2 + 1)^3 \, dx$

10 Find the area enclosed between the curve $y = \ln x$, the y-axis and the lines $y = 1$ and $y = 3$.

11 Find the area bounded by the curve $y = x^2$, the x-axis and the lines $x = -1$ and $x = 2$.

12 Find $\int \sin x° \, dx$.

13 Find the area enclosed between the curves $y = x^2$ and $y = 2 - x^2$.

14 Find the indefinite integral of:

a e^{4x} **b** $\dfrac{x}{x^2 - 9}$ **c** e^{-x}

d $\dfrac{1}{x + 4}$ **e** $(x - 3)(x^2 - 6x + 1)^8$

15 Evaluate $\int_1^2 \dfrac{3x^4 - 2x^3 + x^2 - 1}{x^2} \, dx$.

16 Find the exact area in the first quadrant bounded by $x^2 + y^2 = 9$, the y-axis and the lines $y = 0$ and $y = 3$.

17 Find the area bounded by the curve $y = x^3$, the y-axis and the lines $y = 0$ and $y = 1$.

18 Find the integral of $(7x + 3)^{11}$.

19 Find the area bounded by the curve $y = x^2 - x - 2$, the x-axis and the lines $x = 1$ and $x = 3$.

20 Find the exact area bounded by the curve $y = e^{2x}$, the x-axis and the lines $x = 2$ and $x = 5$.

21 Use the trapezoidal rule with 4 strips to find the area bounded by the curve $y = \ln (x^2 - 1)$, the x-axis and the lines $x = 3$ and $x = 5$.

22 Evaluate $\int_0^4 (3t^2 - 6t + 5) \, dt$.

23 Find the indefinite integral of:

a $\sin 2x$ **b** $3 \cos x$ **c** $\sec^2 5x$ **d** $1 + \sin x$

24 Find the area bounded by the curve $y = x^2 + 2x - 15$ and the x-axis.

ISBN 9780170413220

25 The rate at which a metal cools is given by $R = -16e^{-0.4t}$ degrees min^{-1}. If the temperature is initially 215°C, find:

a the equation for the temperature T of the metal

b the temperature, to the nearest degree, of the metal after:

i 5 minutes **ii** half an hour

26 Evaluate:

a $\int_0^{\frac{\pi}{4}} \cos x \, dx$ **b** $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \sec^2 x \, dx$

27 Find:

a $\int 5(2x-1)^4 \, dx$ **b** $\int \frac{3x^5}{4} \, dx$

28 Find the exact area bounded by the curve $y = \sin x$, the x-axis and the lines $x = \frac{\pi}{4}$ and $x = \frac{\pi}{2}$.

29 Find:

a $\int x^2(x^3-2)^5 \, dx$ **b** $\int x(5x^2+2)^4 \, dx$

c $\int 5x^3(2x^4-1)^2 \, dx$ **d** $\int (x+2)(x^2+4x-3)^3 \, dx$

30 Find the area bounded by the curve $y = \cos 2x$, the x-axis and the lines $x = 0$ and $x = \pi$.

31 Find (in exact form) the approximate area bounded by the curve $y = \sqrt{x-2}$, the x-axis and the line $x = 4$, using:

a a triangle **b** 2 inner rectangles **c** 2 outer rectangles

32 Find $f(x)$ given $f'(x)$ and a point on the graph of $f(x)$.

a $f'(x) = 3x(2x^2-1)^4$ and passing through (1, 3)

b $f'(x) = \sec^2 2x$ and passing through $\left(\frac{\pi}{6}, \frac{\sqrt{3}}{2}\right)$

c $f'(x) = e^{5x}$ and passing through $(0, \frac{1}{5})$

d $f'(x) = x^3(x^4-15)^3$ and passing through (2, 0)

e $f'(x) = \frac{3x^3}{x^4+1}$ and passing through (0, 2)

33 The velocity of a particle is given by $v = \frac{t^2}{\sqrt{t^3+9}}$ m s^{-1}. If the initial displacement is –2 m, find:

a the equation for displacement

b the displacement after 5 s

c when the displacement is 10 m

ISBN 9780170413220

6. CHALLENGE EXERCISE

1 **a** Show that $f(x) = x^3 + x$ is an odd function.

b Hence find the value of $\int_{-2}^{2} f(x)\,dx$.

c Find the total area between $y = f(x)$, the x-axis and the lines $x = -2$ and $x = 2$.

2 **a** Show that $\sec x \operatorname{cosec} x = \dfrac{\sec^2 x}{\tan x}$.

b Hence, or otherwise, find the exact value of $\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \operatorname{cosec} x \sec x\,dx$.

3 Find the area enclosed between the curves $y = (x-1)^2$ and $y = 5 - x^2$.

4 Find the exact value of $\int_{\frac{\pi}{12}}^{\frac{\pi}{8}} \sec^2 2x\,dx$.

5 Evaluate $\int_0^1 \dfrac{x}{(3x^2-4)^2}\,dx$.

6 Use the trapezoidal rule with 4 subintervals to find the area enclosed between the curve $y = \dfrac{3}{x-2}$, the y-axis and the lines $y = 1$ and $y = 3$.

7 **a** Sketch the curve $y = x(x-1)(x+2)$.

b Find the total area enclosed between the curve and the x-axis.

8 Find the area bounded by the parabola $y = x^2$ and the line $y = 4 - x$ correct to 2 decimal places.

9 **a** Find the derivative of $x\sqrt{x+3}$.

b Hence find $\int \dfrac{x+2}{\sqrt{x+3}}\,dx$.

10 **a** Find $\dfrac{d}{dx}(x^2 \ln x)$.

b Hence find the exact value of $\int_1^3 2x(1+2\ln x)\,dx$.

11 Find the area enclosed between the curves $y = \sqrt{x}$ and $y = x^3$.

12 **a** Find the sum of 50 terms of the sequence $2^0, 2^{0.2}, 2^{0.4}, 2^{0.6}, \ldots$

b Hence use 50 inner rectangles to find the approximate area under the curve $y = 2^x$ between $x = 0$ and $x = 10$.

c Find this approximate area by using 100 outer rectangles.

ISBN 9780170413220