3.

TRIGONOMETRIC FUNCTIONS

In this chapter you will study the effect of transformations on trigonometric functions and solve trigonometric equations graphically and algebraically.

CHAPTER OUTLINE

3.01 Transformations of trigonometric functions
3.02 Combined transformations of trigonometric functions
3.03 Trigonometric equations



IN THIS CHAPTER YOU WILL:

- apply and understand the effect of different transformations on trigonometric functions
- solve trigonometric equations graphically and algebraically

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TERMINOLOGY

amplitude: The height from the centre of a sine or cosine function to the maximum or minimum values (peaks and troughs of its graph respectively). For $y = k \sin ax$ and $y = k \cos ax$, the amplitude is k.

centre: The mean value of a sine or cosine function that is equidistant from the maximum and minimum values. For $y = k \sin ax + c$ and $y = k \cos ax + c$, the centre is c.

period: The length of one cycle of a periodic function on the x-axis, before the function repeats itself. For $y = k \sin ax$ and $y = k \cos ax$, the period is $\frac{2\pi}{a}$.

phase: A horizontal shift (translation). For $y = k \sin [a(x + b)]$ and $y = k \cos [a(x + b)]$, the phase is b; that is, the graphs of $y = k \sin ax$ and $y = k \cos ax$ respectively are shifted b units to the left.

WS Transforming trigonometric functions

WS Sketching periodic functions - amplitude and period

WS Sketching periodic functions - phase and vertical shift

3.01 Transformations of trigonometric functions

The transformations you studied in Chapter 2, *Transformations of functions*, can be applied to the trigonometric functions.

Vertical dilations

A vertical dilation of $y = f(x)$ is $y = kf(x)$ with scale factor k.

EXAMPLE 1

a Describe the transformation if $y = \cos x$ is transformed to $y = 3 \cos x$.

b Sketch the graph of $y = 3 \cos x$ in the domain $[0, 2\pi]$ and state its range.

Solution

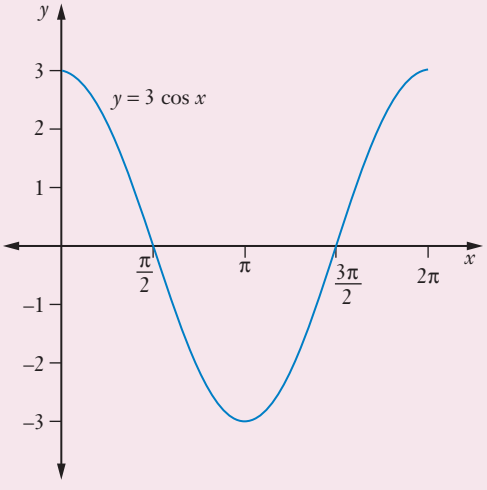
a $y = 3 \cos x$ is a vertical dilation of $y = \cos x$ with scale factor 3.

b A vertical dilation multiplies the y values by 3.

$(0, 1)$	becomes	$(0, 3)$
$\left(\frac{\pi}{2}, 0\right)$	becomes	$\left(\frac{\pi}{2}, 0\right)$
$(\pi, -1)$	becomes	$(\pi, -3)$
$\left(\frac{3\pi}{2}, 0\right)$	becomes	$\left(\frac{3\pi}{2}, 0\right)$
$(2\pi, 1)$	becomes	$(2\pi, 3)$

The range is $[-3, 3]$.

Notice that $y = 3 \cos x$ has amplitude 3.



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Amplitude as a vertical dilation

$y = k \sin x$ or $y = k \cos x$ has **amplitude** k (a vertical dilation with scale factor k).

- If $k > 1$, the function is stretched.
- If $0 < k < 1$, the function is compressed.
- If $k = -1$, the function is reflected in the x-axis.

Note: $y = k \tan x$ does not have an amplitude.

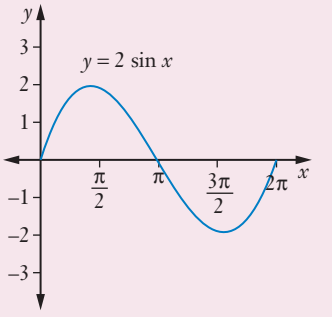
EXAMPLE 2

a Sketch the graph of $f(x) = 2 \sin x$ in the domain $[0, 2\pi]$.

b Find an equation for a cosine function reflected in the x-axis with amplitude 7.

Solution

a This is a vertical dilation of $f(x) = \sin x$ with scale factor 2 (it has amplitude 2).



b $y = \cos x$ is reflected in the x-axis with scale factor $k = -1$.

$y = -\cos x$

Amplitude: $k = 7$

So $y = -7 \cos x$

Horizontal dilations

A horizontal dilation of $y = f(x)$ is $y = f(ax)$ with scale factor $\frac{1}{a}$.

EXAMPLE 3

a Describe the transformation if $f(x) = \sin x$ is transformed to $y = \sin 2x$.

b Draw the graph of $f(x) = \sin 2x$ in the domain $[0, 2\pi]$.

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Solution

a $f(x) = \sin 2x$ is a horizontal dilation of $f(x) = \sin x$ with scale factor $\frac{1}{2}$.

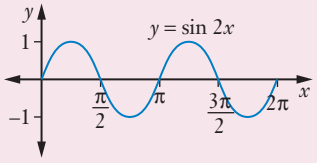
b A horizontal dilation multiplies the x values by $\frac{1}{2}$ (or divides them by 2).

$(0, 0)$	becomes	$(0, 0)$
$\left(\frac{\pi}{2}, 1\right)$	becomes	$\left(\frac{\pi}{4}, 1\right)$
$(\pi, 0)$	becomes	$\left(\frac{\pi}{2}, 0\right)$
$\left(\frac{3\pi}{2}, -1\right)$	becomes	$\left(\frac{3\pi}{4}, -1\right)$
$(2\pi, 0)$	becomes	$(\pi, 0)$

Notice that these image points lie in the domain $[0, \pi]$ and not $[0, 2\pi]$.

The **period** of $y = \sin 2x$ is $\frac{2\pi}{2} = \pi$.

To sketch the function in the domain $[0, 2\pi]$ we repeat the sine curve from $x = \pi$ to 2π.



Notice that a horizontal dilation compresses the graph of $y = \sin x$, which changes its period. The function $y = \sin 2x$ has 2 complete sine function cycles in the domain $[0, 2\pi]$.

Period as a horizontal dilation

$y = \sin ax$ has **period** $\frac{2\pi}{a}$.

$y = \cos ax$ has **period** $\frac{2\pi}{a}$.

$y = \tan ax$ has **period** $\frac{\pi}{a}$.

- If $a > 1$, the function is compressed horizontally.
- If $0 < a < 1$, the function is stretched horizontally.
- If $a = -1$, the function is reflected in the y-axis.

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EXAMPLE 4

a Find the period of each function.

i $y = \cos x$ **ii** $f(x) = \sin 5x$ **iii** $y = \tan 2x$

b Sketch each graph in the domain $[0, 2\pi]$.

i $y = \tan \frac{x}{2}$ **ii** $y = \sin(-x)$

Solution

a **i** $y = \cos x$ has period 2π.

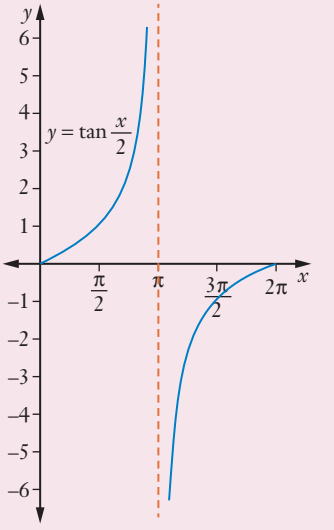
ii $f(x) = \sin 5x$ has period $\frac{2\pi}{5}$.

iii $y = \tan 2x$ has period $\frac{\pi}{2}$.

b **i** $y = \tan \frac{x}{2}$ is a horizontal dilation of $y = \tan x$.

It has period $\frac{\pi}{\frac{1}{2}}$ or 2π.

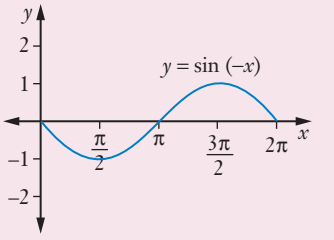
So there will be one cycle of the tan function in the domain $[0, 2\pi]$.



ii $y = \sin(-x)$ is a reflection of $y = \sin x$ in the y-axis, so $a = -1$.

This will change the x values. Transforming points in the domain $[-2\pi, 0]$ will give image points in the domain $[0, 2\pi]$.

$(0, 0)$	becomes	$(0 \times -1, 0) \equiv (0, 0)$
$\left(-\frac{\pi}{2}, -1\right)$	becomes	$\left(\frac{\pi}{2}, -1\right)$
$(-\pi, 0)$	becomes	$(\pi, 0)$
$\left(-\frac{3\pi}{2}, 1\right)$	becomes	$\left(\frac{3\pi}{2}, 1\right)$
$(-2\pi, 0)$	becomes	$(2\pi, 0)$



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Vertical translations

A vertical translation of $y = f(x)$ is $y = f(x) + c$.

EXAMPLE 5

Sketch the graph of $y = \cos x + 2$ in the domain $[-\pi, \pi]$.

Solution

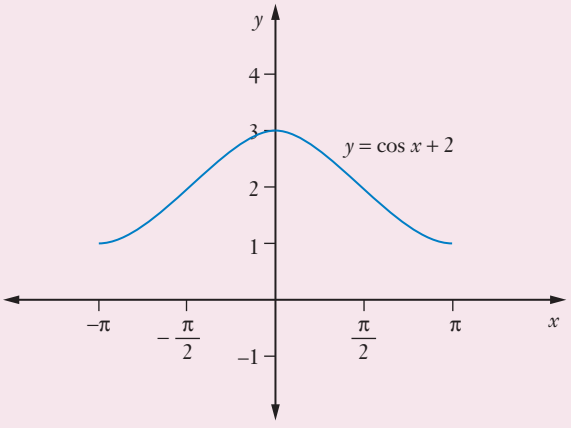
$y = f(x) + c$ is a vertical translation of $y = f(x)$.

So $y = \cos x + 2$ is a vertical translation of $y = \cos x$ up 2 units.

This changes the y values by adding 2 to each.

The domain is $[-\pi, \pi]$.

$(-\pi, -1)$	becomes	$(-\pi, -1 + 2) \equiv (-\pi, 1)$
$\left(-\frac{\pi}{2}, 0\right)$	becomes	$\left(-\frac{\pi}{2}, 2\right)$
$(0, 1)$	becomes	$(0, 3)$
$\left(\frac{\pi}{2}, 0\right)$	becomes	$\left(\frac{\pi}{2}, 2\right)$
$(\pi, -1)$	becomes	$(\pi, 1)$



Notice that the centre of the function is 2.

Centre as a vertical translation

The **centre** of $y = \sin x + c$ and $y = \cos x + c$ is c.

- If $c > 0$, the centre is translated upwards.
- If $c < 0$, the centre is translated downwards.

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EXAMPLE 6

a Find the centre of the function:

i $f(x) = \sin x - 7$ **ii** $y = \cos x + 4$

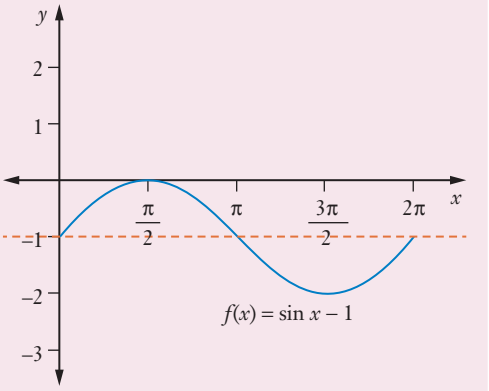
b Sketch the graph of $f(x) = \sin x - 1$.

Solution

a **i** The centre is -7.

ii The centre is 4.

b This is a vertical translation 1 unit down of $f(x) = \sin x$. The centre is -1.



Horizontal translations

A horizontal translation of $y = f(x)$ is given by $y = f(x + b)$.

EXAMPLE 7

Sketch the graph of $y = \sin\left(x - \frac{\pi}{2}\right)$ in the domain $[0, 2\pi]$.

Solution

$y = \sin\left(x - \frac{\pi}{2}\right)$ is a horizontal translation of $y = \sin x$ by $\frac{\pi}{2}$ units to the right.

We change the x values by adding $\frac{\pi}{2}$.

But since we need the *transformed* values of x to be in the domain $[0, 2\pi]$, our *original* values need to be in the domain $\left[0 - \frac{\pi}{2}, 2\pi - \frac{\pi}{2}\right] \equiv \left[-\frac{\pi}{2}, \frac{3\pi}{2}\right]$.

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$\left(-\frac{\pi}{2}, -1\right)$ becomes $\left(-\frac{\pi}{2}+\frac{\pi}{2}, -1\right) \equiv (0, -1)$

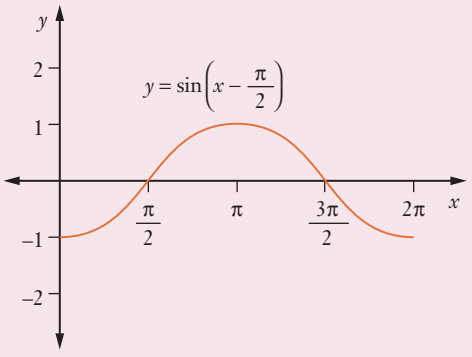
$(0, 0)$ becomes $\left(0+\frac{\pi}{2}, 0\right) \equiv \left(\frac{\pi}{2}, 0\right)$

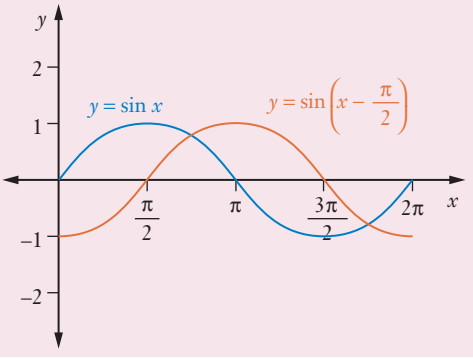
$\left(\frac{\pi}{2}, 1\right)$ becomes $\left(\frac{\pi}{2}+\frac{\pi}{2}, 1\right) \equiv (\pi, 1)$

$(\pi, 0)$ becomes $\left(\pi+\frac{\pi}{2}, 0\right) \equiv \left(\frac{3\pi}{2}, 0\right)$

$\left(\frac{3\pi}{2}, -1\right)$ becomes $\left(\frac{3\pi}{2}+\frac{\pi}{2}, -1\right) \equiv (2\pi, -1)$

Instead of finding points, you could sketch $y = \sin x$ and then shift it $\frac{\pi}{2}$ units to the right.





Phase as a horizontal translation

The **phase** of $y = \sin(x + b)$, $y = \cos(x + b)$ and $y = \tan(x + b)$ is b.

- If $b > 0$, the phase shift is to the left.
- If $b < 0$, the phase shift is to the right.

Phase shift of trigonometric functions

EXAMPLE 8

a Explain the meaning of $\frac{\pi}{4}$ in the equation $y = \tan\left(x + \frac{\pi}{4}\right)$.

b Sketch the graph of $y = \tan\left(x + \frac{\pi}{4}\right)$ in the domain $[0, 2\pi]$.

Solution

a The function $y = \tan\left(x + \frac{\pi}{4}\right)$ has a phase of $\frac{\pi}{4}$ (to the left).

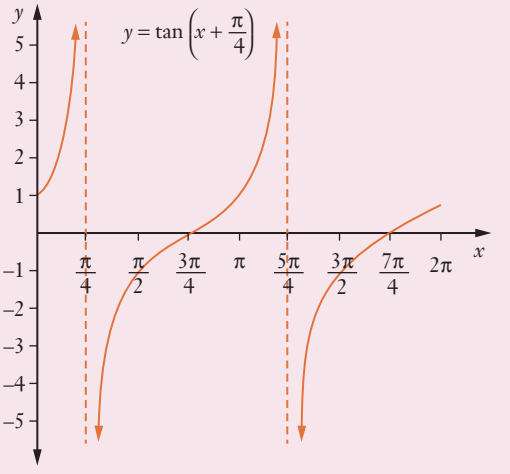
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b To find points on the transformed graph, subtract $\frac{\pi}{4}$ from x values.

But since we need the *transformed* values of x to be in the domain $[0, 2\pi]$,

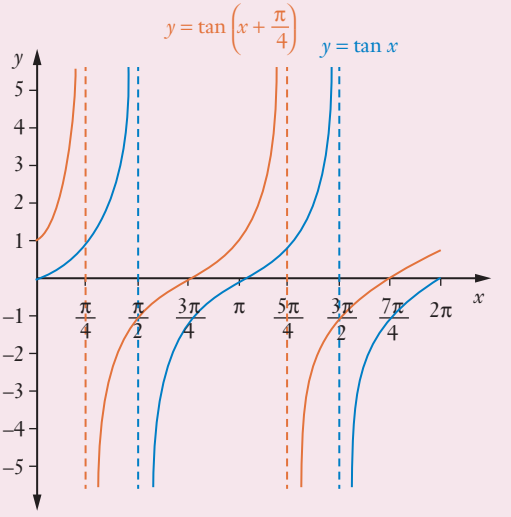
our *original* values need to be in the domain $\left[0+\frac{\pi}{4}, 2\pi+\frac{\pi}{4}\right] \equiv \left[\frac{\pi}{4}, \frac{9\pi}{4}\right]$.

$\left(\frac{\pi}{4}, 1\right)$	becomes	$(0, 1)$
Undefined at $x = \frac{\pi}{2}$	becomes	Undefined at $x = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}$
$\left(\frac{3\pi}{4}, -1\right)$	becomes	$\left(\frac{\pi}{2}, -1\right)$
$(\pi, 0)$	becomes	$\left(\frac{3\pi}{4}, 0\right)$
$\left(\frac{5\pi}{4}, 1\right)$	becomes	$(\pi, 1)$
Undefined at $x = \frac{3\pi}{2}$	becomes	Undefined at $x = \frac{3\pi}{2} - \frac{\pi}{4} = \frac{5\pi}{4}$
$\left(\frac{7\pi}{4}, -1\right)$	becomes	$\left(\frac{3\pi}{2}, -1\right)$
$(2\pi, 0)$	becomes	$\left(\frac{7\pi}{4}, 0\right)$
$\left(\frac{9\pi}{4}, -1\right)$	becomes	$(2\pi, 1)$



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Instead of finding points, you could sketch $y = \tan x$ and then shift it $\frac{\pi}{4}$ units to the left.



Exercise 3.01 Transformations of trigonometric functions

1 Describe whether each transformation of a trigonometric function changes its amplitude, period, centre or phase.

a horizontal translation **b** vertical dilation

c horizontal dilation **d** vertical translation

2 Sketch the graph of each function in the domain $[0, 2\pi]$.

a $y = 5 \sin x$ **b** $f(x) = 2 \tan x$ **c** $y = -\cos x$

d $f(x) = -2 \sin x$ **e** $y = -\tan x$

3 Sketch the graph of each function in the domain $[0, 2\pi]$.

a $y = \sin x + 1$ **b** $y = \tan x - 2$ **c** $f(x) = \cos x - 3$

4 Sketch the graph of each function in the domain $[0, 2\pi]$.

a $y = \cos 4x$ **b** $y = \sin \frac{x}{2}$

c $f(x) = \tan 2x$ **d** $y = \tan \frac{x}{4}$

5 Sketch the graph of each function in the domain $[0, 2\pi]$.

a $y = \cos (x + \pi)$

b $y = \tan\left(x - \frac{\pi}{2}\right)$

c $y = \sin\left(x - \frac{\pi}{4}\right)$

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6 Find the equation of the transformation of $y = \sin x$ if the transformed function has:

a amplitude 9

b a reflection in the x-axis

c centre -4

d period π

e a phase shift of π units to the right

7 Find the equation of the transformation of $y = \cos x$ if the transformed function has:

a amplitude 4

b a phase of $\frac{\pi}{3}$ units

c centre 8

d period $\frac{\pi}{2}$

e a vertical dilation with scale factor 7

8 Find the equation of the transformation of $y = \tan x$ if the transformed function has:

a period 2π

b a shift of $\frac{\pi}{6}$ units to the right

c a reflection in the y-axis

9 Sketch each graph in the domain $[-\pi, \pi]$.

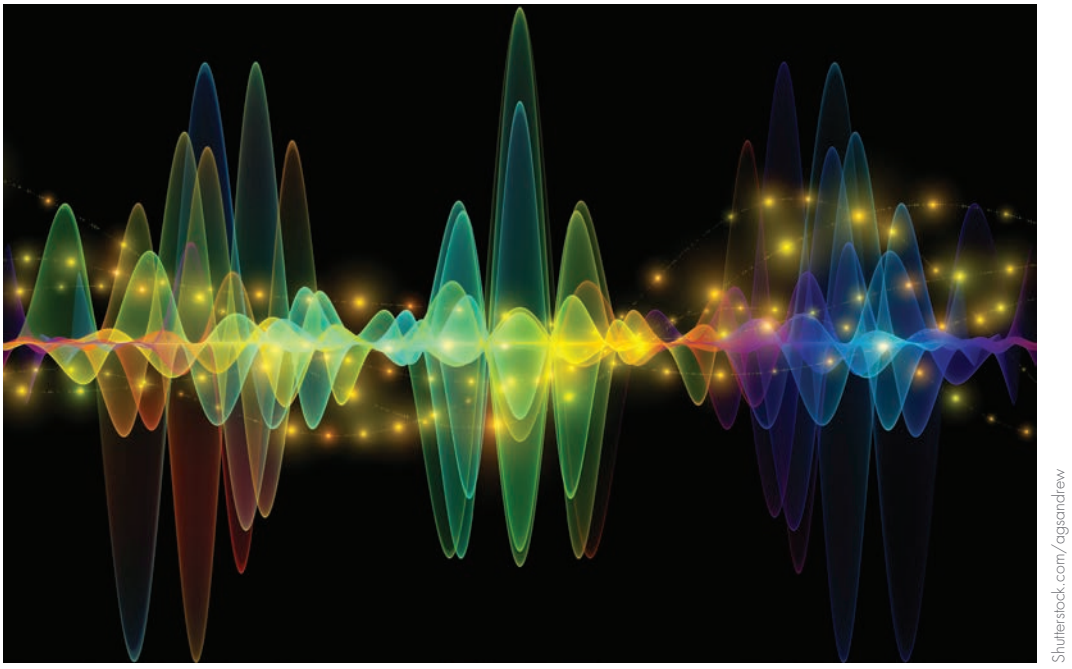
a $y = 3 \sin x$

b $y = \tan(-x)$

c $f(x) = \cos 2x$

d $y = \sin(x - \pi)$

e $f(x) = -\cos x$



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WS Graphing trigonometric functions

WS Sketching periodic functions

3.02 Combined transformations of trigonometric functions

We can put all the information about trigonometric functions together.

General equation of trigonometric functions

Function	Amplitude	Period	Phase	Centre
$y = k\sin[a(x+b)]+c$	k	$\frac{2\pi}{a}$	b Shift left if $b>0$ Shift right if $b<0$	c Shift up if $c>0$ Shift down if $c<0$
$y = k\cos[a(x+b)]+c$	k	$\frac{2\pi}{a}$		
$y = k\tan[a(x+b)]+c$	No amplitude	$\frac{\pi}{a}$		

EXAMPLE 9

a Sketch each function in the domain $[0, 2\pi]$.

i $y = 4\sin\frac{x}{2}+1$ **ii** $y = 3\cos\left(x-\frac{\pi}{4}\right)$

b Find the equation of a cosine function that has amplitude 5, period 4π, centre -2 and a phase of 2 units to the left.

Solution

a **i** $y = 4\sin\frac{x}{2}+1$ has amplitude 4,

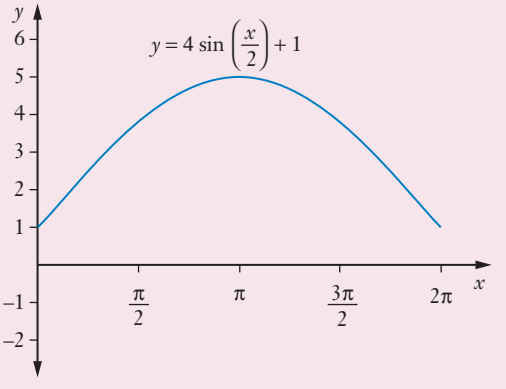
period $\frac{2\pi}{\frac{1}{2}} = 4\pi$ and centre 1.

Period 4π means only half the sine function curve will be in the domain $[0, 2\pi]$.

Centre 1 and amplitude 4 means:

Minimum $1-4=-3$

Maximum $1+4=5$



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ii $y = 3\cos\left(x - \frac{\pi}{4}\right)$ has amplitude 3, period 2π, phase $\frac{\pi}{4}$ to the right and centre 0.

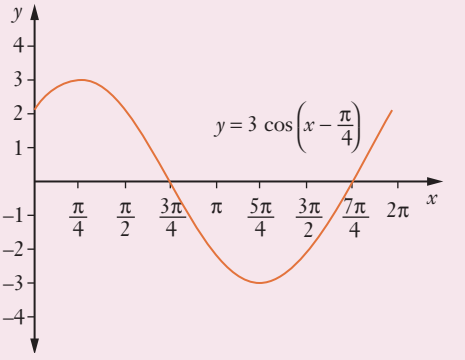
A phase is a horizontal translation, so it changes the x values and the domain.

Add $\frac{\pi}{4}$ to each x value.

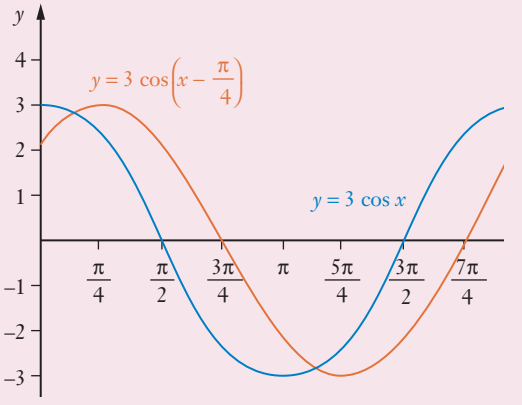
But since we need the *transformed* values of x to be in the domain $[0, 2\pi]$,

our *original* values need to be in the domain $\left[0 - \frac{\pi}{4}, 2\pi - \frac{\pi}{4}\right] \equiv \left[-\frac{\pi}{4}, \frac{7\pi}{4}\right]$.

$\left(-\frac{\pi}{4}, 2.1\right)$	becomes	$(0, 2.1)$
$(0, 3)$	becomes	$\left(\frac{\pi}{4}, 3\right)$
$\left(\frac{\pi}{4}, 2.1\right)$	becomes	$\left(\frac{\pi}{2}, 2.1\right)$
$\left(\frac{\pi}{2}, 0\right)$	becomes	$\left(\frac{3\pi}{4}, 0\right)$
$\left(\frac{3\pi}{4}, -2.1\right)$	becomes	$(\pi, -2.1)$
$(\pi, -3)$	becomes	$\left(\frac{5\pi}{4}, -3\right)$
$\left(\frac{5\pi}{4}, -2.1\right)$	becomes	$\left(\frac{3\pi}{2}, -2.1\right)$
$\left(\frac{3\pi}{2}, 0\right)$	becomes	$\left(\frac{7\pi}{4}, 0\right)$
$\left(\frac{7\pi}{4}, 2.1\right)$	becomes	$(2\pi, 2.1)$



Alternatively, sketch $y = 3\cos x$ and then shift it $\frac{\pi}{4}$ units to the right.



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b $y = k \cos [a(x + b)] + c$.

Amplitude: $k = 5$

Period: $\frac{2\pi}{a} = 4\pi$

$2\pi = 4\pi a$

$\frac{1}{2} = a$

Phase: $b = 2$

Centre: $c = -2$

The equation is $y = 5 \cos\left[\frac{1}{2}(x + 2)\right] - 2$

Exercise 3.02 Combined transformations of trigonometric functions

1 Sketch the graph of each function in the domain $[0, 2\pi]$.

a $y = 2 \sin x - 3$

b $y = -\tan 2x$

c $f(x) = \cos\left(x + \frac{\pi}{2}\right) + 1$

d $y = \sin\left(-\frac{x}{2}\right) + 2$

e $f(x) = 3 \cos 2x - 2$

2 **a** Find the equation of the transformed function if $y = \sin x$ is vertically dilated with scale factor 5, horizontally dilated with scale factor $\frac{1}{3}$, vertically translated 6 units down and horizontally translated 5 units to the left.

b Describe each transformation as a change in period, amplitude, centre or phase of the function.

3 Find the equation of the transformed function of $y = \cos x$ if it is:

a vertically dilated with scale factor 4, horizontally dilated with scale factor $\frac{1}{6}$, vertically translated 2 units up and horizontally translated $\frac{\pi}{3}$ units to the right

b reflected in the x-axis, reflected in the y-axis, translated 5 units down and π units to the left.

4 Sketch each graph in the domain $[-\pi, \pi]$.

a $y = 3 \sin 2x$

b $y = 2 \tan \frac{x}{2} + 1$

c $f(x) = -2 \cos 3x$

d $y = 5 \sin x - 3$

e $y = \cos(-2x) + 1$

5 Describe the features of each function in terms of amplitude, period, centre, and phase.

a $y = 3 \tan 4x - 5$

b $y = 8 \cos(x + \pi) - 3$

c $y = 5 \sin [2(x - 3)] + 1$

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6 Find the equation of each function.

a a sine function with amplitude 7, period π, phase of 1 unit to the right and centre -3

b a cosine function with amplitude 1, a reflection in the x-axis, period $\frac{2\pi}{5}$ and centre 2

c a tangent function with period 2π, a reflection in the x-axis and a phase of 2 units to the left

d $y = \sin x$ with a vertical dilation scale factor 4, a reflection in the y-axis, a horizontal dilation scale factor 3, a vertical translation 2 units up and a horizontal translation 5 units to the left

7 Describe the features of $y = k \operatorname{cosec}[a(x + b)] + c$.

8 Find the equation of the transformed function if $y = \tan x$ is translated 3 units to the right and then dilated horizontally with scale factor $\frac{1}{4}$.

9 The water depth at a harbour entrance is 5 m at low tide and 25 m at high tide. The time between each low tide is around 12 hours.

a Find the centre of the tidal motion.

b What is the amplitude and period?

c Write an equation for the water depth D metres in terms of time t hours as a cosine function.

10 Find an equation for blood pressure, B, as a sine function of time, t minutes, if the maximum blood pressure is 120 and the minimum is 80, with a heart rate of 60 beats per minute.



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WS
Solving trigonometric equations graphically

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Trigonometric equations

3.03 Trigonometric equations

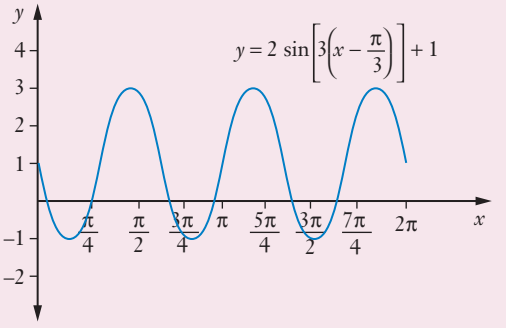
Graphical solutions

We can use the work on transformations to help solve trigonometric equations graphically.

EXAMPLE 10

a The graph of the trigonometric function $y = 2\sin\left[3\left(x - \frac{\pi}{3}\right)\right] + 1$ is shown for $[0, 2\pi]$.

Find the number of solutions to the trigonometric equation $2\sin\left[3\left(x - \frac{\pi}{3}\right)\right] + 1 = 0$ for $[0, 2\pi]$.



b **i** Sketch the graphs of $y = \frac{x}{4} - 1$ and $y = 3\cos x - 2$ for $[0, 2\pi]$.

ii Find the number of solutions to the equation $3\cos x - 2 = \frac{x}{4} - 1$ for $[0, 2\pi]$.

iii Solve the equation graphically.

Solution

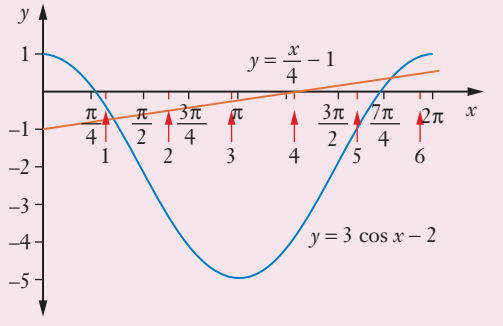
a To solve $2\sin\left[3\left(x - \frac{\pi}{3}\right)\right] + 1 = 0$ graphically, we find the x-intercepts.

The function has 6 x-intercepts in the domain $[0, 2\pi]$.

So the equation has 6 solutions.

b **i** $y = 3\cos x - 2$ has amplitude 3 and centre -2.

$y = \frac{x}{4} - 1$ is a linear function with x-intercept 4 and y-intercept -1.



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ii The solutions to $3\cos x - 2 = \frac{x}{4} - 1$ are shown by where the graphs $y = 3\cos x - 2$ and $y = \frac{x}{4} - 1$ intersect.

The graphs intersect in 2 places in $[0, 2\pi]$ so the equation has 2 solutions.

iii The graphs intersect just after $x = 1$ and $x = \frac{7\pi}{4} \approx 5.5$.

A precise graph drawn on graph paper or using technology would show that the solutions are $x \approx 1.1$ and 5.6.

Algebraic solutions

EXAMPLE 11

Solve for $[0°, 360°]$:

a $\sin 3x = \frac{1}{2}$ **b** $\cos(2x - 60°) = \frac{1}{\sqrt{2}}$

Solution

a For the domain $[0°, 360°]$

$0° \leq x \leq 360°$

$0° \leq 3x \leq 1080°$

So the new domain is $[0°, 1080°]$ (3 revolutions of the circle)

$\sin 3x = \frac{1}{2}$ sin x > 0 in 1st and 2nd quadrants

$3x = 30°, 180° - 30°, 360° + 30°, 360° + (180° - 30°), 720° + 30°, 720° + (180° - 30°)$

$= 30°, 150°, 390°, 510°, 750°, 870°$

$x = 10°, 50°, 130°, 170°, 250°, 290°$

b $0° \leq x \leq 360°$

$0° \leq 2x \leq 720°$

$-60° \leq 2x - 60° \leq 660°$

So the new domain is $[-60°, 660°]$ (2 revolutions of the circle starting at $-60°$)

$\cos(2x - 60°) = \frac{1}{\sqrt{2}}$ cos x > 0 in 1st and 4th quadrants

$2x - 60° = -45°, 45°, 360° - 45°, 360° + 45°$

($720° - 45°$ is outside the domain)

$= -45°, 45°, 315°, 405°$

$2x = 15°, 105°, 375°, 465°$

$x = 7.5°, 52.5°, 187.5°, 232.5°$

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EXAMPLE 12

Solve each equation for $[0, 2\pi]$.

a $6\cos 2x - 3 = 0$ **b** $\tan\left(x - \frac{\pi}{4}\right) = \sqrt{3}$

Solution

a For the domain $[0, 2\pi]$

$0 \leq x \leq 2\pi$

$0 \leq 2x \leq 4\pi$, so when solving for $2x$ we need to go around the circle twice.

$6\cos 2x - 3 = 0$

$6\cos 2x = 3$

$\cos 2x = \frac{1}{2}$ cos $x > 0$ in 1st and 4th quadrants

$2x = \frac{\pi}{3}, 2\pi - \frac{\pi}{3}, 2\pi + \frac{\pi}{3}, 2\pi + 2\pi - \frac{\pi}{3}$ as $0 \leq 2x \leq 4\pi$

$= \frac{\pi}{3}, \frac{5\pi}{3}, \frac{7\pi}{3}, \frac{11\pi}{3}$

$x = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}$

b $0 \leq x \leq 2\pi$

$-\frac{\pi}{4} \leq x - \frac{\pi}{4} \leq \frac{7\pi}{4}$

So the new domain is $\left[-\frac{\pi}{4}, \frac{7\pi}{4}\right]$ $\left(1 \text{ revolution of the circle starting at } -\frac{\pi}{4}\right)$

$\tan\left(x - \frac{\pi}{4}\right) = \sqrt{3}$ tan $x > 0$ in 1st and 3rd quadrants

$x - \frac{\pi}{4} = \frac{\pi}{3}, \pi + \frac{\pi}{3}$

$= \frac{\pi}{3}, \frac{4\pi}{3}$

$x = \frac{\pi}{3} + \frac{\pi}{4}, \frac{4\pi}{3} + \frac{\pi}{4}$

$= \frac{7\pi}{12}, \frac{19\pi}{12}$

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Exercise 3.03 Trigonometric equations

1 By drawing the graph of $y = 2 \sin 3x$ in the domain $[0, 2\pi]$:

a find the number of solutions of $2 \sin 3x = 1$

b solve $2 \sin 3x = 1$ graphically

2 **a** Sketch the graphs of $y = -\cos x + 3$ and $y = x - 1$ for $[0, 2\pi]$.

b Solve:

i $-\cos x + 3 = x - 1$ **ii** $-\cos x + 3 = 2$

3 Solve for $[0°, 360°]$:

a $2 \sin 2x = 1$ **b** $\tan 3x = -1$ **c** $\cos (x + 90°) = \frac{\sqrt{3}}{2}$

d $\tan (x - 45°) = \sqrt{3}$ **e** $\sin (x + 60°) = 0$

4 Solve for $[0, 2\pi]$:

a $\tan 2x = \sqrt{3}$ **b** $2 \cos 3x + 1 = 0$ **c** $4 \sin^2\left(x - \frac{\pi}{3}\right) = 3$

d $2 \cos^2 2x - 1 = 0$ **e** $\cos (x + \pi) = 1$

5 Solve for $[-\pi, \pi]$:

a $\tan 3x = 1$ **b** $\cos (x + \frac{\pi}{4}) = \frac{1}{\sqrt{2}}$ **c** $\sin 2x = -1$

d $\cos (x - \frac{\pi}{2}) = 0$ **e** $\tan^2 4x = 0$

6 Solve for $[0, 2\pi]$:

a $\cos 2\left(x - \frac{\pi}{2}\right) = \frac{1}{2}$ **b** $2 \sin\left(3x + \frac{3\pi}{2}\right) = 1$

7 The function $T = 15 \cos \frac{\pi t}{6} + 20$ models the average monthly temperatures in Nelson Springs, starting in January.

a Find the amplitude, period and centre of the function.

b Solve $15 \cos \frac{\pi t}{6} + 20 = 35$ and explain the meaning of the solutions.

8 A set of tidal waves has a maximum height of 20 m and a minimum height of 6 m. The waves break every 10 seconds.

a Find the equation of a sine function that describes the motion of the waves.

b Find the first 4 times that the waves reach their maximum height.

c Find the first time that the waves reach their minimum height.

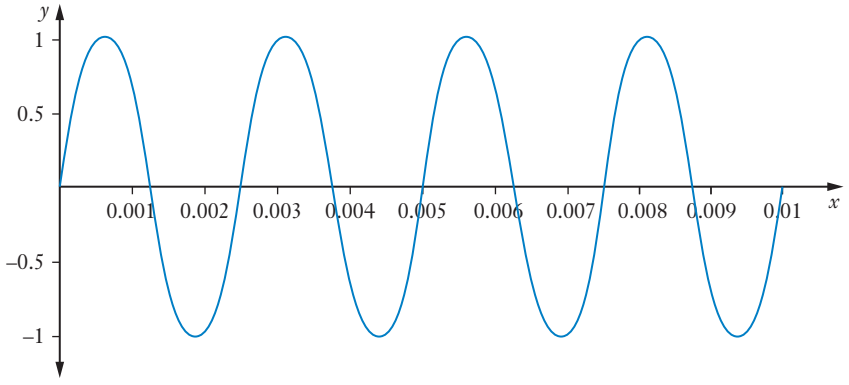
d When will the height of the waves be in the centre?

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9 Sound waves have the shape of sine functions. The graph below shows the sound wave that occurs when playing the note A above middle C on a piano. Its equation is $y = \sin(880\pi x)$ where x is time in seconds.



a Find the amplitude and period of this sound wave.

b Use the graph to solve for [0, 0.01]:

i $\sin(880\pi x) = 0.5$ **ii** $\sin(880\pi x) = 0$

c Solve algebraically for [0, 0.01] (to 2 significant figures):

i $\sin(880\pi x) = 0.5$ **ii** $\sin(880\pi x) = 0$

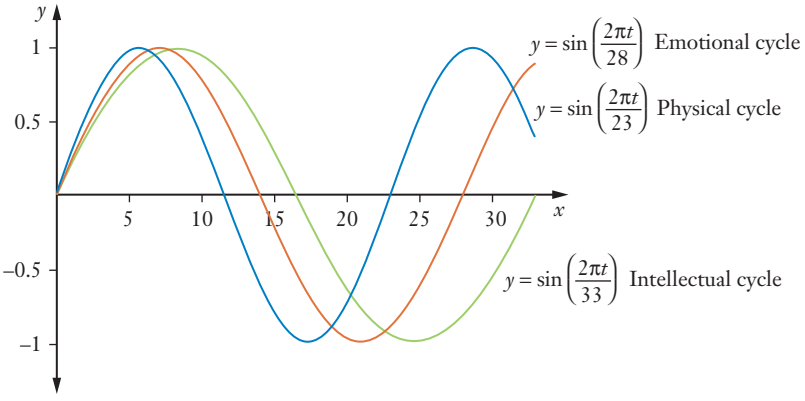
d The higher the amplitude, the more volume the sound has (it is louder). (The word 'amplifier' comes from this property.) Find the equation of the note A that is 3 times as loud as the one drawn.

e A note that is higher in pitch has a higher frequency (more cycles) than a lower note. Draw a rough sketch of a middle C note with the same volume as the A note above C.

f The unit of measurement for frequency is hertz (Hz), the number of wave cycles of a sound in 1 second. What is the frequency in hertz of note A?

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10 Biorhythms is a theory that emotional, physical and mental activity in humans can be modelled by 3 sine functions: physical $y = \sin\frac{2\pi t}{23}$, emotional $y = \sin\frac{2\pi t}{28}$ and intellectual $y = \sin\frac{2\pi t}{33}$ where t is time in days, starting from your date of birth. It was first developed by German doctor Wilhelm Fliess in 1878 but became popular in the 1970s when computers were able to chart the 3 biorhythms. Their graphs are sketched below.



a What is the period of each function? What does this mean?

b When do the physical and intellectual graphs intersect?

c When do the emotional and physical graphs intersect?

d Biorhythms are supposed to be at optimal levels when $y = 1$ (maximum points). Estimate the range of times when all 3 biorhythms are near optimal levels together.

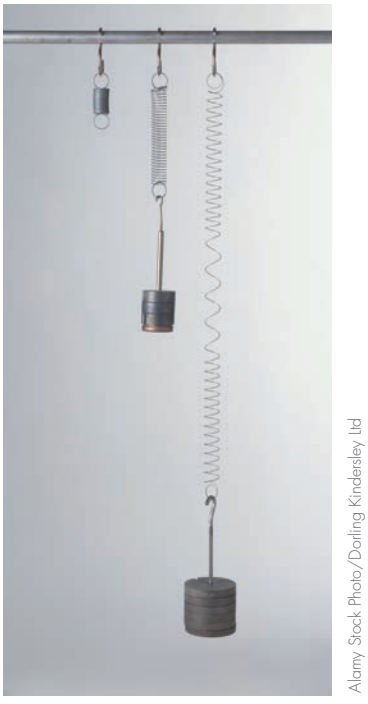
11 A vertical spring is pulled down and then let go. It bounces back up and down again according to the equation $h = 12\cos t + 15$ where h is the height of the spring in cm and t is time in seconds.

a Describe the significance of the 15 in the equation.

b What are the maximum and minimum heights of the spring?

c What is the height of the spring after π seconds?

d At what times will the spring be at its minimum height?



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3. TEST YOURSELF

WS
Formula sheet: Functions, Trigonometry and Vectors

Qz
Practice quiz

For Questions **1** to **3**, choose the correct answer **A**, **B**, **C** or **D**.

1 The function $y = 2\cos 3x - 7$ has:

A amplitude 2, period 3 and centre -7

B amplitude 7, period $\frac{1}{3}$ and centre 2

C amplitude 2, period $\frac{2\pi}{3}$ and centre 7

D amplitude 2, period $\frac{2\pi}{3}$ and centre -7

2 The equation of a function with phase π units to the left is:

A $y = \tan(x + \pi)$ **B** $y = \tan(\pi x)$

C $y = \tan x + \pi$ **D** $y = \tan(x - \pi)$

3 The solution of $\cos 2x = 1$ in the domain $[0, 2\pi]$ is:

A $x = 0, 2\pi$ **B** $x = 0, \pi, 2\pi$

C $x = \frac{\pi}{2}, \frac{3\pi}{2}$ **D** $x = \pi$

4 Sketch the graph of each function in the domain $[0, 2\pi]$.

a $y = 3\cos x$ **b** $y = \tan\frac{x}{2}$

c $y = \sin x - 2$ **d** $y = -\sin x$

e $y = 3\cos 2x - 1$

5 The function $h = 3\cos\left(\frac{2\pi t}{3}\right) + 10$ shows the water level in a lock in metres over time t hours.

a Find the maximum and minimum levels of water and when they occur.

b Solve $3\cos\left(\frac{2\pi t}{3}\right) + 10 = 11$ and explain what the solution means.



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6 Simplify:

a $2\cot^2 x + 2$

b $\tan A \operatorname{cosec} A$

c $(\sec A + \tan A)(\sec A - \tan A)$

d $\sin(180° - x)$

7 Solve for $[0, 2\pi]$:

a $4\cos^2 x = 3$

b $2\sin 2x = 1$

c $\cos\left(x - \frac{\pi}{2}\right) = -1$

d $\tan^2\left(x + \frac{\pi}{6}\right) = 3$

8 A person's blood pressure has a maximum pressure of 135 and a minimum pressure of 85 and the heartbeat is 70 beats per minute.

a Write an equation showing this blood pressure y as a sine function of time t minutes.

b Draw a graph showing this function.

9 Solve for $[-\pi, \pi]$:

a $2\cos 2x = 1$

b $\tan\left(x - \frac{\pi}{4}\right) = -\frac{1}{\sqrt{3}}$

c $\sin\left(x + \frac{\pi}{2}\right) = \frac{1}{\sqrt{2}}$

10 The high tide mark on a cliff above a river is 80 m and the low tide mark is 50 m. The average time between high tides in the river is approximately 13 hours.

a Write the equation for the height of the tides as a sine function.

b Find the times when the river is halfway between high and low tides.

11 Find the equation of each transformed function of $y = \cos x$ that has:

a period π

b amplitude 5

c a reflection in the x-axis

d a phase of $\frac{\pi}{6}$ units to the right

e centre 4

12 **a** Sketch the graph of $y = 2\sin\frac{x}{2} - 1$ for $[0, 2\pi]$.

b From the graph, solve $2\sin\frac{x}{2} - 1 = 0$.

c Solve $2\sin\frac{x}{2} - 1 = 0$ algebraically.

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13 State the amplitude, period, centre and phase of each function.

a $y = 2 \sin 3x - 1$

b $y = \cos\left(\frac{x}{2} + \pi\right)$

c $y = -3 \tan\left(5x - \frac{\pi}{4}\right)$

14 Solve for $[0°, 360°]$:

a $\tan(x + 45°) = 1$

b $\sqrt{2} \cos(x - 20°) + 1 = 0$

c $3 \sin[2(x + 10°)] - 2 = 0$

15 Solve for $[-180°, 180°]$:

a $5 \tan 2x = -5$

b $\cos[3(x - 30°)] + 1 = 0$

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3. CHALLENGE EXERCISE

1 **a** Find the amplitude, period and phase of the function $y = 2\cos\left(2x - \frac{\pi}{2}\right)$.

b Solve $2\cos\left(2x - \frac{\pi}{2}\right) = \sqrt{3}$ for $[0, 2\pi]$.

2 Find the equation of:

a a cosine function with amplitude 8, period 2π and centre 4

b a sine function with amplitude 2, period $\frac{\pi}{4}$, phase $\frac{\pi}{3}$ units to the right and centre 3

c a tangent function with period 2π and phase $\frac{\pi}{2}$ units to the left

3 Sketch the graph of $y = 3\sec 2x$ for $[0, 2\pi]$.

4 Solve $\operatorname{cosec} 2x = \sqrt{2}$ for $[0, 2\pi]$.

5 **a** Find the equation of the transformation of $y = \cos x$ if it has a vertical translation 5 units down, a horizontal translation $\frac{\pi}{6}$ units to the right, then a vertical stretch, scale factor 4 and a horizontal stretch, scale factor 3.

b Find the amplitude, period, centre and phase of this transformed function.

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