Knox Grammar School

2024

Name: _____

Trial Higher School Certificate

Examination

Teacher: _____

Year 12 Extension 2

Mathematics

General Instructions

- Reading time – 10 minutes
- Working time – 3 hours
- Write using blue or black pen only
- Board approved calculators may be used
- A reference sheet is provided at the back of this paper in addition to the separate yellow document. You may use either.
- In Questions 11-16, show relevant mathematical reasoning and/or calculations.

Teachers:

Mr Bradford (Examiner)

Mr Menzies

Section I ~ Pages 1-5

- 10 marks
- Attempt Questions 1 – 10
- Allow about 15 minutes for this section

Section II ~ Pages 7-12

- 90 marks
- Attempt Questions 11 – 16
- Allow about 2 hours 45 minutes for this section

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Section I
Questions 1 – 10 (1 mark for each question)

Read each question and choose an answer A, B, C or D.
Record your answer on the Answer Sheet provided.
Allow about 15 minutes for this section.

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1 Let $\alpha = 2 - i$ and $\beta = 1 - 3i$.

What is the value of $(\alpha - \beta)^4$?

A) $7 - 24i$

B) $-7 - 24i$

C) $-7 - 40i$

D) $41 - 24i$

2 Consider the statement.

"If a quadrilateral is a rectangle, then the diagonals of the quadrilateral are equal in length."

Which of the following is true about this statement and its converse?

A) Both the statement and its converse are true.

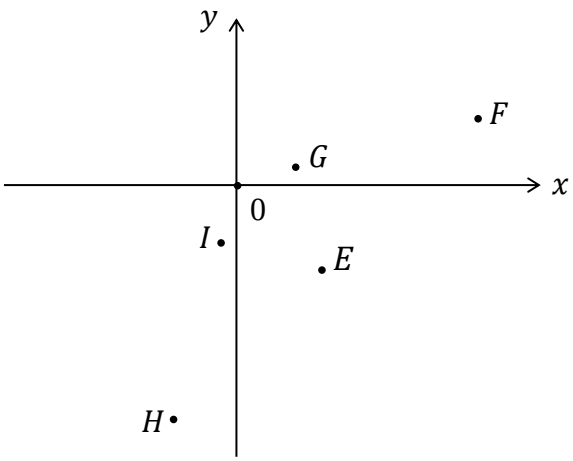
B) Both the statement and its converse are false.

C) The statement is false and its converse is true.

D) The statement is true and its converse is false.

3 In the Argand diagram the point E represents a complex number.

When this number is multiplied by $\left(\sqrt{3}+i\right)^4\left(\frac{1}{4}-\frac{\sqrt{3}}{4}i\right)^3$ it gives a new complex number.



Which of the following points represents the new complex number?

A) F

B) G

C) H

D) I

4 A particle is moving in simple harmonic motion. The displacement x metres of the particle is given by $x = 1 + 4\sin\left(2t - \frac{\pi}{5}\right)$, where t is measured in seconds.

Which of the following is the first time the velocity of the particle is $-8\ ms^{-1}$?

A) $\frac{3\pi}{5}$

B) $\frac{\pi}{10}$

C) $\frac{7\pi}{10}$

D) $\frac{2\pi}{5}$

5 Let the points A and B represent the complex numbers $\lambda = ln^3\alpha + i\ ln^2\ \alpha$ and $\phi = -2\ ln\,\alpha + 8i$ respectively.

When the point B is rotated about the origin by $\frac{\pi}{2}$ in a clockwise direction we get the point C.

For which values of α do A and C coincide?

A) $\alpha = 1$

B) $\alpha = e^2$

C) $\alpha = 1\ or\ \alpha = e^2$

D) $\alpha = e\ \ or\ \ \alpha = e^2$

6 Consider the position vector of a particle $\underset{\sim}{r} = -3\sin t\ \underset{\sim}{i} + 3\cos t\ \underset{\sim}{j} + t\ \underset{\sim}{k}$.

Which of the following statements best describes the motion of the particle?

A) A spiral about the z axis in an anticlockwise direction.

B) A spiral about the z axis in a clockwise direction.

C) A spiral about the x axis in a anticlockwise direction.

D) A spiral about the x axis in a clockwise direction.

7 Consider the following statement:

"$\forall\ n \in \mathbb{N}$, if n^2 is odd then n is odd."

Which of the following is the negation of the statement?

A) $\forall\ n \in \mathbb{N}$, if n^2 is not odd then n is also not odd.

B) $\exists\ n \in \mathbb{N}$, such that n is not odd and n^2 is not odd.

C) $\forall\ n \in \mathbb{N}$, if n is not odd then n^2 is not odd.

D) $\exists\ n \in \mathbb{N}$, such that n^2 is odd and n is not odd.

8 The equations of two lines ℓ_1 and ℓ_2 are

$$\overrightarrow{r_1} = \begin{pmatrix} -1 + t\,(2^{\lambda}) \\ -5 + t\,(6 - 2^{\lambda}) \\ -2 + t\,(7 - 2^{\lambda}) \end{pmatrix} \text{ and } \overrightarrow{r_2} = \begin{pmatrix} 1 + t\,(8 + 2^{\lambda}) \\ 2 + 6t \\ -3 + t\,(5 + 2^{\lambda}) \end{pmatrix},$$

where t is a real number and λ is a constant.

For what value of λ are the two lines ℓ_1 and ℓ_2 parallel?

A) 4

B) 3

C) 2

D) 1

9 A particle initially at rest at the origin starts to move along the x axis.

Its velocity at any time t in seconds is $v\ m\ s^{-1}$.

The acceleration of the particle is given by $a = (1 + v^2)^{\frac{3}{2}}$.

Which of the following is the correct expression for the velocity v ?

A) $v = \tan(\cos^{-1} t)$

B) $v = \tan(\sin^{-1} t)$

C) $v = \sin^{-1}(\tan t)$

D) $v = \cos(\sin^{-1} t)$

10 The complex number z satisfies $|z + a| = a$, where a is a positive real number.

The greatest distance that z can be from the point P representing the complex number $ka + a\ (k + 1)i$, where k is a positive real number, is $\left(3\sqrt{2} + 1\right)a$.

What is the value of k ?

A) 1

B) 2

C) 3

D) 4

End of Section I

MATHEMATICS EXTENSION 2

Section II

90 marks

Attempt Questions 11 – 16

Allow about 2 hours and 45 minutes for this section.

INSTRUCTIONS

- Answer each question in a separate writing booklet.
- Extra writing booklets are available.
- Your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) Use a SEPARATE writing booklet

a) Consider the complex numbers $\alpha = 2i$ and $\beta = \sqrt{3} + i$. **3**

Show that $(\alpha - \beta)^6$ is purely real.

b) Consider the line ℓ with equation $\underset{\sim}{r} = \begin{pmatrix} 3 \\ -2 \\ 5 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 6 \\ 4 \end{pmatrix}$. **2**

Show that the point $A\ (2, -5, 3)$ lies on the line ℓ.

c) Solve $z^2 - z + (4 - 2i) = 0$. Give your answers in the form $x + yi$, where x and y are real. **3**

d) *PQRS* is a parallelogram with vertices $P\ (2, 4, 3)$, $Q\ (4, 3, 1)$, $R\ (3, 2, 5)$ and $S\ (x, y, z)$. **3**

Find the coordinates of S and the length of the diagonal $\left|\overrightarrow{QS}\right|$.

e) A particle is moving along the x axis in simple harmonic motion centred at $x = 2$ and with amplitude 3. **4**

The velocity of the particle is given by $v^2 = p + qx - 6x^2$, where p and q are constants.

Find the period of the motion.

Question 12 (15 marks) Use a SEPARATE writing booklet

a) Prove that $x^3 + \frac{1}{x^3} \geq 2$, where $x > 0$. **2**

b) Consider the predicate statement $P(n)$ where $n \in \mathbb{N}$. **3**

"If $5n^4 + 8n$ is multiple of 16, then *n* is even."

Prove that the converse of P is true.

c) Find all the 4th roots of $1 + \sqrt{3}\,i$. Give your answers in exponential form. **3**

d) A particle starts moving from $x = ln\,(e^4 + 1)$ with velocity e and **3**

acceleration given by $v^2\,(e^4 + \ln v\,)$, where v is the velocity of the particle.

Find an expression for x, the displacement of the particle, in terms of v.

e) The polynomial $Q(z) = z^4 - 8z^3 + pz^2 + qz - 80$ has a root $3 + i$, where p and q are real numbers.

i) Find all the roots of $Q(z)$. **3**

ii) Write $Q(z)$ as a product of two real quadratic factors. **1**

Question 13 (15 marks) Use a SEPARATE writing booklet

a) The cubic equation $2x^3 - 7x + 4 = 0$ has roots α, β and γ. **3**

Find a cubic equation with roots $\frac{1}{\alpha\beta}$, $\frac{1}{\alpha\gamma}$ and $\frac{1}{\beta\gamma}$.

b) A boat needs to travel from port A to port B on a true bearing of 333° at a speed of 35 kmh^{-1}. **3**

However, a constant current of 15 kmh^{-1} is flowing from the southwest.

Find the speed and the direction to which the boat should head to compensate for the current and maintain its speed of 35 kmh^{-1} in the desired direction.

c) Use a contrapositive proof to prove that " If p is divisible by 5 then p can be expressed as a sum of five consecutive integers". **2**

d) Find $\int \left(\frac{(2\tan\theta + 3)\sec^2\theta}{\sec^2\theta + \tan\theta} \right) d\theta$ **3**

e) i) Show $\frac{2k}{k+2} < \frac{2k+2}{k+3}$ for $k > 0$. **1**

ii) Use mathematical induction to prove that **3**

$$\frac{1}{3!} + \frac{2}{4!} + \frac{3}{5!} + \cdots + \frac{n}{(n+2)!} < \frac{2n}{n+2} - \frac{1}{(n+2)!} \text{ for } n \geq 1.$$

Question 14 (14 marks) Use a SEPARATE writing booklet

a) The complex equation $z^5 = i$ has roots are $e^{\frac{i\pi}{10}}$, $e^{\frac{9i\pi}{10}}, e^{\frac{-7i\pi}{10}}, e^{\frac{-3i\pi}{10}}$ and α.

i) Determine the value of α. **1**

ii) On an Argand diagram, sketch the roots of this equation. **2**

iii) Show that $\cos\frac{\pi}{5}\sin\frac{\pi}{10} = \frac{1}{4}$. **3**

b) A particle of mass 1 kg is projected from the origin with initial speed $V\ ms^{-1}$ at an angle α to the horizontal plane.

The forces acting on the particle are gravity $\overrightarrow{1g}$ and air resistance $5\vec{v}$, where $\vec{v}$ is the velocity vector of the particle.

Let the acceleration due to gravity be $10\ ms^{-2}$.

The position vector of the particle, at time t seconds after the particle is projected, is $\vec{r}(t)$ and the velocity vector is $\vec{v}(t)$.

i) Show that $\vec{v}(t) = \begin{pmatrix} Ve^{-5t}\cos\alpha \\ (V\sin\alpha + 2)\,e^{-5t} - 2 \end{pmatrix}$. **3**

ii) Given that $\overrightarrow{v}(1) = \begin{pmatrix} 250\,e^{-5} \\ \left(250\sqrt{3} + 2\right)e^{-5} - 2 \end{pmatrix}$, find the initial speed V and the angle of projection α. **2**

iii) Show that the ratio of the horizontal velocity at the origin to the horizontal velocity at the maximum height is $\left(1 + 125\sqrt{3}\right) : 1$. **3**

Question 15 (15 marks) Use a SEPARATE writing booklet

a) A particle of mass m kg is moving along the x axis under the action of a resisting force $m\,(pv + v^2)$, where v is its velocity and p is a positive constant.

Initially, the particle is at $x = \ln 2$ and is travelling with velocity p.

i) Show that the displacement x of the particle in terms of v is **2**

$$x = \ln\left(\frac{4p}{p + v}\right).$$

ii) Show that the time t, which has elapsed when the particle is travelling with velocity v, is **3**

$$t = \frac{1}{p}\ln\left(\frac{p + v}{2v}\right).$$

iii) It took the particle $\frac{1}{2}\ln 2$ seconds to reach the point where $x = \ln 3$. Find the value of p. **2**

b) Two lines, L_1 and L_2, are represented by position vectors $\underset{\sim}{r_1}$ and $\underset{\sim}{r_2}$ respectively.

The vector equations of these two lines are

$$\underset{\sim}{r_1} = \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} \quad \text{and} \quad \underset{\sim}{r_2} = \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix},$$

where λ and μ are real numbers.

i) Show that L_1 and L_2 are skew lines. **4**

ii) Find the shortest distance between the two skew lines L_1 and L_2. **4**

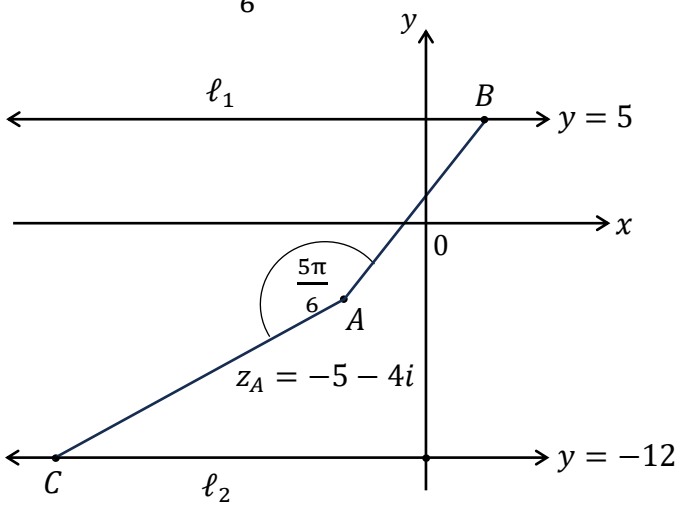
Question 16 (16 marks) Use a SEPARATE writing booklet

a) The point A, which represents the complex number $z_A = -5 - 4i$, lies between two parallel lines ℓ_1 and ℓ_2.

The equations of ℓ_1 and ℓ_2 are $y = 5$ and $y = -12$ respectively.

The point $B(b, 5)$, which represents the complex number z_B, lies on ℓ_1.
The point $C(c, -12)$, which represents the complex number z_C, lies on ℓ_2 such that $AC = 2AB$ and $\angle \text{BAC} = \frac{5\pi}{6}$, as shown below.

Not to Scale



Find the exact value of the complex number z_C. **4**

b) Let $I_n = \int_0^{\pi/2} \sin^{2n+3} x \cos^5 x dx, n = 0,1,2,\cdots$

i) Prove that $I_n = \frac{n+1}{n+4} I_{n-1}, \ for\ n \geq 1.$ **3**

ii) Deduce that $I_n = \frac{1}{(n+4)(n+3)(n+2)}$. **3**

iii) Let $J_n = \int_0^1 x^{4n+7}(1-x^4)^2 \, dx, \ n = 0, 1, 2, \ldots$

Show that $J_n = \frac{1}{2} I_n$. **3**

c) Prove by contradiction that

$(\sin x)^{\frac{2}{n}} + (\cos x)^{\frac{2}{n}} \geq 1, for\ 0 \leq x \leq \frac{\pi}{2}$ and $n = 3, 4, 5, \ldots.$ **3**

END OF PAPER

NSW Education Standards Authority

2024 HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics Advanced
Mathematics Extension 1
Mathematics Extension 2

REFERENCE SHEET

Measurement

Length

$$l = \frac{\theta}{360} \times 2\pi r$$

Area

$$A = \frac{\theta}{360} \times \pi r^2$$

$$A = \frac{h}{2}(a + b)$$

Surface area

$$A = 2\pi r^2 + 2\pi rh$$

$$A = 4\pi r^2$$

Volume

$$V = \frac{1}{3}Ah$$

$$V = \frac{4}{3}\pi r^3$$

Functions

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

For $ax^3 + bx^2 + cx + d = 0$:

$$\alpha + \beta + \gamma = -\frac{b}{a}$$

$$\alpha\beta + \alpha\gamma + \beta\gamma = \frac{c}{a}$$

$$\text{and } \alpha\beta\gamma = -\frac{d}{a}$$

Relations

$$(x - h)^2 + (y - k)^2 = r^2$$

Financial Mathematics

$$A = P(1 + r)^n$$

Sequences and series

$$T_n = a + (n - 1)d$$

$$S_n = \frac{n}{2}[2a + (n - 1)d] = \frac{n}{2}(a + l)$$

$$T_n = ar^{n-1}$$

$$S_n = \frac{a(1 - r^n)}{1 - r} = \frac{a(r^n - 1)}{r - 1}, \ r \neq 1$$

$$S = \frac{a}{1 - r}, \ |r| < 1$$

Logarithmic and Exponential Functions

$$\log_a a^x = x = a^{\log_a x}$$

$$\log_a x = \frac{\log_b x}{\log_b a}$$

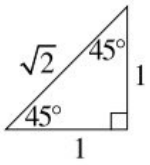
$$a^x = e^{x \ln a}$$

Trigonometric Functions

$$\sin A = \frac{\text{opp}}{\text{hyp}}, \quad \cos A = \frac{\text{adj}}{\text{hyp}}, \quad \tan A = \frac{\text{opp}}{\text{adj}}$$

$$A = \frac{1}{2}ab\sin C$$

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

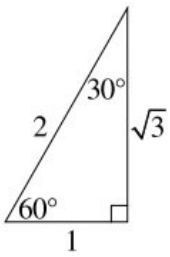


$$c^2 = a^2 + b^2 - 2ab\cos C$$

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

$$l = r\theta$$

$$A = \frac{1}{2}r^2\theta$$



Trigonometric identities

$$\sec A = \frac{1}{\cos A},\ \cos A \neq 0$$

$$\operatorname{cosec} A = \frac{1}{\sin A},\ \sin A \neq 0$$

$$\cot A = \frac{\cos A}{\sin A},\ \sin A \neq 0$$

$$\cos^2 x + \sin^2 x = 1$$

Compound angles

$$\sin(A + B) = \sin A\cos B + \cos A\sin B$$

$$\cos(A + B) = \cos A\cos B - \sin A\sin B$$

$$\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A\tan B}$$

If $t = \tan\frac{A}{2}$ then $\sin A = \frac{2t}{1+t^2}$

$$\cos A = \frac{1 - t^2}{1 + t^2}$$

$$\tan A = \frac{2t}{1 - t^2}$$

$$\cos A\cos B = \frac{1}{2}\left[\cos(A - B) + \cos(A + B)\right]$$

$$\sin A\sin B = \frac{1}{2}\left[\cos(A - B) - \cos(A + B)\right]$$

$$\sin A\cos B = \frac{1}{2}\left[\sin(A + B) + \sin(A - B)\right]$$

$$\cos A\sin B = \frac{1}{2}\left[\sin(A + B) - \sin(A - B)\right]$$

$$\sin^2 nx = \frac{1}{2}(1 - \cos 2nx)$$

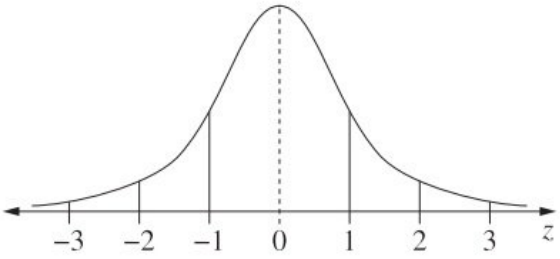
$$\cos^2 nx = \frac{1}{2}(1 + \cos 2nx)$$

Statistical Analysis

$$z = \frac{x - \mu}{\sigma}$$

An outlier is a score
less than $Q_1 - 1.5 \times IQR$
or
more than $Q_3 + 1.5 \times IQR$

Normal distribution



- approximately 68% of scores have z-scores between –1 and 1
- approximately 95% of scores have z-scores between –2 and 2
- approximately 99.7% of scores have z-scores between –3 and 3

$$E(X) = \mu$$

$$\operatorname{Var}(X) = E\left[(X - \mu)^2\right] = E\left(X^2\right) - \mu^2$$

Probability

$$P(A \cap B) = P(A)P(B)$$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$P(A|B) = \frac{P(A \cap B)}{P(B)},\ P(B) \neq 0$$

Continuous random variables

$$P(X \leq x) = \int_a^x f(x)\,dx$$

$$P(a < X < b) = \int_a^b f(x)\,dx$$

Binomial distribution

$$P(X = r) = {}^nC_r p^r (1 - p)^{n-r}$$

$$X \sim \text{Bin}(n, p)$$

$$\Rightarrow \quad P(X = x) = \binom{n}{x} p^x (1 - p)^{n-x},\ x = 0, 1, \ldots, n$$

$$E(X) = np$$

$$\operatorname{Var}(X) = np(1 - p)$$

Differential Calculus

Function	Derivative
$y = f(x)^n$	$\frac{dy}{dx} = nf'(x)[f(x)]^{n-1}$
$y = uv$	$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$
$y = g(u)$ where $u = f(x)$	$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$
$y = \frac{u}{v}$	$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$
$y = \sin f(x)$	$\frac{dy}{dx} = f'(x)\cos f(x)$
$y = \cos f(x)$	$\frac{dy}{dx} = -f'(x)\sin f(x)$
$y = \tan f(x)$	$\frac{dy}{dx} = f'(x)\sec^2 f(x)$
$y = e^{f(x)}$	$\frac{dy}{dx} = f'(x)e^{f(x)}$
$y = \ln f(x)$	$\frac{dy}{dx} = \frac{f'(x)}{f(x)}$
$y = a^{f(x)}$	$\frac{dy}{dx} = (\ln a)f'(x)a^{f(x)}$
$y = \log_a f(x)$	$\frac{dy}{dx} = \frac{f'(x)}{(\ln a)f(x)}$
$y = \sin^{-1} f(x)$	$\frac{dy}{dx} = \frac{f'(x)}{\sqrt{1-[f(x)]^2}}$
$y = \cos^{-1} f(x)$	$\frac{dy}{dx} = -\frac{f'(x)}{\sqrt{1-[f(x)]^2}}$
$y = \tan^{-1} f(x)$	$\frac{dy}{dx} = \frac{f'(x)}{1+[f(x)]^2}$

Integral Calculus

$$\int f'(x)[f(x)]^n dx = \frac{1}{n+1}[f(x)]^{n+1} + c \quad \text{where } n \neq -1$$

$$\int f'(x)\sin f(x)\,dx = -\cos f(x) + c$$

$$\int f'(x)\cos f(x)\,dx = \sin f(x) + c$$

$$\int f'(x)\sec^2 f(x)\,dx = \tan f(x) + c$$

$$\int f'(x)e^{f(x)}dx = e^{f(x)} + c$$

$$\int \frac{f'(x)}{f(x)}dx = \ln|f(x)| + c$$

$$\int f'(x)a^{f(x)}dx = \frac{a^{f(x)}}{\ln a} + c$$

$$\int \frac{f'(x)}{\sqrt{a^2-[f(x)]^2}}dx = \sin^{-1}\frac{f(x)}{a} + c$$

$$\int \frac{f'(x)}{a^2+[f(x)]^2}dx = \frac{1}{a}\tan^{-1}\frac{f(x)}{a} + c$$

$$\int u\frac{dv}{dx}dx = uv - \int v\frac{du}{dx}dx$$

$$\int_a^b f(x)dx$$

$$\approx \frac{b-a}{2n}\left\{f(a) + f(b) + 2\left[f(x_1) + \cdots + f(x_{n-1})\right]\right\}$$

where $a = x_0$ and $b = x_n$

Combinatorics

$${}^nP_r = \frac{n!}{(n-r)!}$$

$$\binom{n}{r} = {}^nC_r = \frac{n!}{r!(n-r)!}$$

$$(x+a)^n = x^n + \binom{n}{1}x^{n-1}a + \cdots + \binom{n}{r}x^{n-r}a^r + \cdots + a^n$$

Vectors

$$\left|\underset{\sim}{u}\right| = \left|x\underset{\sim}{i} + y\underset{\sim}{j}\right| = \sqrt{x^2+y^2}$$

$$\underset{\sim}{u}\cdot\underset{\sim}{v} = \left|\underset{\sim}{u}\right|\left|\underset{\sim}{v}\right|\cos\theta = x_1x_2 + y_1y_2,$$

where $\underset{\sim}{u} = x_1\underset{\sim}{i} + y_1\underset{\sim}{j}$

and $\underset{\sim}{v} = x_2\underset{\sim}{i} + y_2\underset{\sim}{j}$

$$\underset{\sim}{r} = \underset{\sim}{a} + \lambda\underset{\sim}{b}$$

Complex Numbers

$$z = a + ib = r(\cos\theta + i\sin\theta)$$
$$= re^{i\theta}$$

$$\left[r(\cos\theta + i\sin\theta)\right]^n = r^n(\cos n\theta + i\sin n\theta)$$
$$= r^n e^{in\theta}$$

Mechanics

$$\frac{d^2x}{dt^2} = \frac{dv}{dt} = v\frac{dv}{dx} = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$$

$$x = a\cos(nt+\alpha) + c$$

$$x = a\sin(nt+\alpha) + c$$

$$\ddot{x} = -n^2(x-c)$$

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