2017 HSC Examination

# Mathematics Extension 2

# SOLUTIONS

Written by Jonathan Le

# Multiple Choice

1. C
2. B
3. D
4. C
5. B
6. A
7. A
8. B
9. C
10. B

## Brief Explanations

**Question 1** It has 8 roots so it is either (C) or (D). Must be (C) because 1 is a root.

**Question 2** $PS = ePM$ but $e = \frac{1}{2}$, so it must be an ellipse.

**Question 3** Trial and error.

**Question 4** All single roots became double roots and the curve is non-negative, hence (C).

**Question 5** Note that $P(\alpha) = \alpha^3 - 2\alpha + 2 = 0$, then do a similar thing for $\beta$ and $\gamma$, then add the three equations.

**Question 6** Use the conjugate root theorem.

**Question 7** Split it as two integrals, then use standard properties of even/odd functions.

**Question 8** Find $f(-x)$ for each option.

**Question 9** $\frac{dy}{dt} = \frac{dy}{dx} \times \frac{dx}{dt}$, then use implicit differentiation to find $\frac{dy}{dx}$. Substitute that in to find that $\frac{dy}{dt} = -x$, so it's either (C) or (D). Test a random point on the first quadrant and find that $\frac{dy}{dt} < 0$ but $\frac{dx}{dt} > 0$, hence (C).

**Question 10** Draw a concave up and increasing function like $y = e^x$ to use as an example and recognise that the formulas in (A) and (B) represent the area of the upper bound trapezium. Hence, being an upper bound, the answer is (B).

# Written Response

**Question 11 (a) (i)**

$$\arg z = -\frac{\pi}{3}$$

**Question 11 (a) (ii)**

$$\begin{aligned}\arg\left(\frac{z}{w}\right) &= \arg(z) - \arg(w) \\ &= -\frac{\pi}{3} - \frac{\pi}{4} \\ &= -\frac{7\pi}{12}\end{aligned}$$

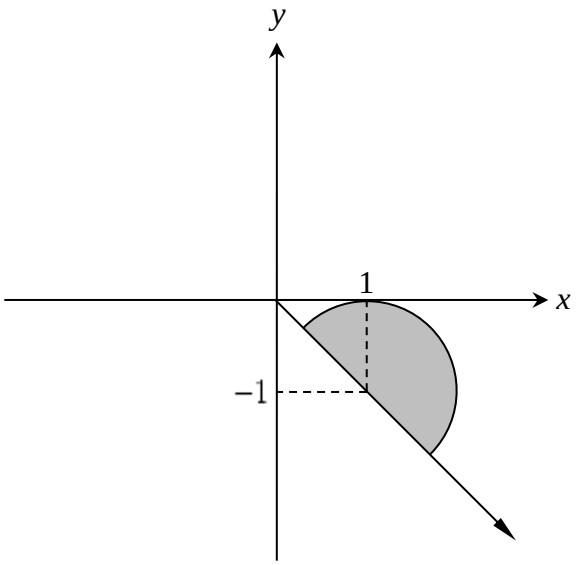
**Question 11 (b)**

$$\begin{aligned}\tan\alpha &= m_{\text{asymptote}} \\ &= \frac{b}{a} \\ &= \frac{2}{2\sqrt{3}} \\ &= \frac{1}{\sqrt{3}}\end{aligned}$$

Hence, $\alpha = \frac{\pi}{6}$.

**Question 11 (c)**

The circle becomes $\left|z-(1-i)\right|\le 1$.




**Question 11 (d)**

$$\begin{aligned}\int_0^{\frac{2\pi}{3}} \frac{1}{1+\cos\theta}\times\frac{1-\cos\theta}{1-\cos\theta}\,d\theta &= \int_0^{\frac{2\pi}{3}} \frac{1-\cos\theta}{\sin^2\theta}\,d\theta \\ &= \int_0^{\frac{2\pi}{3}} \csc^2\theta-\csc\theta\cot\theta\,d\theta \\ &= \left[-\cot\theta+\csc\theta\right]_0^{\frac{2\pi}{3}} \\ &= \frac{1}{\sqrt{3}}+\frac{2}{\sqrt{3}} \\ &= \sqrt{3}\end{aligned}$$

**Question 11 (e)**

First, construct a horizontal strip of thickness $\delta y$ and rotate that about the *x* axis. The cylindrical shell will be oriented such that the axis of symmetry is horizontal.

The height of the cylinder is the difference of the *x* coordinates, which is

$$\begin{aligned} x_2 - x_1 &= (3-y) - \left(\frac{y}{2}\right) \\ &= 3 - \frac{3y}{2} \\ &= \frac{3}{2}(2-y) \end{aligned}$$

The radius is just the *y* coordinate of any point on the strip, which is just *y*.

Hence, the volume can be given by

$$\begin{aligned} V &= 2\pi \int_0^2 y \times \frac{3}{2}(2-y)\,dy \\ &= 3\pi \int_0^2 2y - y^2 dy \end{aligned}$$

**Question 11 (f)**

Compute the differential term $dx = 2\sin\theta\cos\theta\,d\theta$.

$$\begin{aligned} I &= \int_0^{\frac{\pi}{4}} \sqrt{\frac{\sin^2\theta}{1-\sin^2\theta}} \times 2\sin\theta\cos\theta\,d\theta \\ &= 2\int_0^{\frac{\pi}{4}} \sin^2\theta\,d\theta \\ &= \int_0^{\frac{\pi}{4}} 1-\cos 2\theta\,d\theta \\ &= \left[\theta - \frac{1}{2}\sin 2\theta\right]_0^{\frac{\pi}{4}} \\ &= \frac{\pi}{4} - \frac{1}{2} \end{aligned}$$

**Question 12 (a) (i)**

Note that

$$\begin{aligned} f(x) &= \frac{e^x - 1}{e^x + 1} \\ &= \frac{e^x + 1 - 2}{e^x + 1} \\ &= 1 - \frac{2}{e^x + 1} \end{aligned}$$

Hence, we may differentiate without having to use the quotient rule.

$$\begin{aligned} f'(x) &= \frac{d}{dx}\left(1 - \frac{2}{e^x + 1}\right) \\ &= \frac{2e^x}{\left(e^x + 1\right)^2} \\ &> 0 \end{aligned}$$

Hence, the function is increasing.

**Question 12 (a) (ii)**

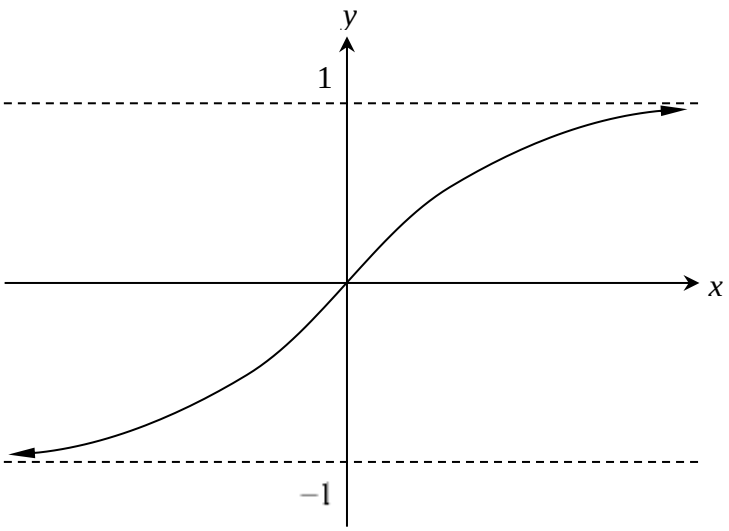
$$\begin{aligned} f(-x) &= \frac{e^{-x} - 1}{e^{-x} + 1} \times \frac{e^x}{e^x} \\ &= \frac{1 - e^x}{1 + e^x} \\ &= -f(x) \end{aligned}$$

**Question 12 (a) (iii)**

Recall that $f(x) = 1 - \frac{2}{e^x + 1}$. From here, it's easy to see that as $x \to \infty$, $f(x) \to 1^-$ since the right term vanishes whilst still remaining positive.

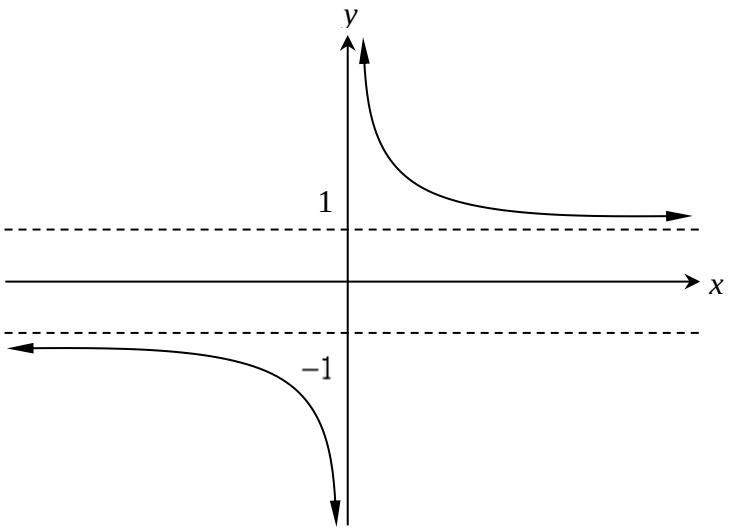
**Question 12 (a) (iv)**

Putting everything together, we obtain the following diagram.




**Question 12 (a) (v)**

We draw the reciprocal of the curve in (iv)



**Question 12 (b)**

Use the quadratic formula

$$\begin{aligned} z &= \frac{-(2+3i) \pm \sqrt{(2+3i)^2 - 4(1+3i)}}{2} \\ &= \frac{-2-3i \pm \sqrt{-9}}{2} \\ &= \frac{-2-3i \pm 3i}{2} \\ &= -1 \text{ or } -1-3i \end{aligned}$$

**Question 12 (c)**

Use integration by parts.

$$\begin{aligned} u &= \tan^{-1} x & dv &= x\,dx \\ du &= \frac{dx}{1+x^2} & v &= \frac{1}{2}x^2 \end{aligned}$$

Hence, we have

$$\begin{aligned} I &= \frac{1}{2}x^2 \tan^{-1} x - \frac{1}{2}\int \frac{x^2}{1+x^2}\,dx \\ &= \frac{1}{2}x^2 \tan^{-1} x - \frac{1}{2}\int 1 - \frac{1}{1+x^2}\,dx \\ &= \frac{1}{2}x^2 \tan^{-1} x - \frac{1}{2}\left(x - \tan^{-1} x\right) + C \\ &= \frac{1}{2}\left(x^2 \tan^{-1}(x) - x + \tan^{-1}(x)\right) + C \end{aligned}$$

**Question 12 (d) (i)**

We know that we can express the polynomial in the form $P(x) = (x-a)^2 Q(x)$, where $Q(x)$ is some polynomial, since we are given that $(x-a)^2$ is a factor. Clearly $P(a) = 0$.

Differentiate $P(x) = (x-a)^2 Q(x)$ using the product rule.

$$\begin{aligned} P'(x) &= 2(x-a)Q(x) + (x-a)^2 Q'(x) \\ &= (x-a)\left(2Q(x) + (x-a)Q'(x)\right) \end{aligned}$$

And again from here, we see clearly that $P'(a) = 0$.

**Question 12 (d) (ii)**

Differentiating and factorising, we have

$$\begin{aligned} P'(x) &= 4x^3 - 9x^2 + 2x \\ &= x\left(4x^2 - 9x + 2\right) \\ &= x(4x-1)(x-2) \end{aligned}$$

Solving this, and using (i) this means that the possible multiple roots are $x = 0,\ \frac{1}{4},\ 2$.

So we substitute each of these into the polynomial to see which one is actually the multiple root. We have $P(0) = 4 \neq 0$, so this cannot be the multiple root. But $P(2) = 0$, so this must be the multiple root, and therefore $\alpha = 2$.

**Question 13 (a)**

Standard proof.

$$\begin{aligned}\left(\sqrt{r}-\sqrt{s}\right)^2 &\geq 0\\ r+s-2\sqrt{rs} &\geq 0\\ \frac{r+s}{2} &\geq \sqrt{rs}\end{aligned}$$

**Question 13 (b) (i)**

Consider the sum of roots in triplets.

$$\begin{aligned}-c &= \alpha\beta\left(\frac{1}{\alpha}\right)+\alpha\left(\frac{1}{\alpha}\right)\left(\frac{1}{\beta}\right)+\alpha\beta\left(\frac{1}{\beta}\right)+\beta\left(\frac{1}{\alpha}\right)\left(\frac{1}{\beta}\right)\\ &= \alpha+\beta+\frac{1}{\alpha}+\frac{1}{\beta}\end{aligned}$$

But this is precisely the sum of roots, which is $-a$.

Hence, $-a=-c$ and thus $a=c$.

**Question 13 (b) (ii)**

Consider the sum of roots in pairs.

$$\begin{aligned}b &= \alpha\left(\frac{1}{\alpha}\right)+\alpha\beta+\frac{\alpha}{\beta}+\frac{\beta}{\alpha}+\frac{1}{\alpha\beta}+\beta\left(\frac{1}{\beta}\right)\\ &= 2+\underbrace{\left(\alpha\beta+\frac{1}{\alpha\beta}\right)}_{\geq 2}+\underbrace{\left(\frac{\alpha}{\beta}+\frac{\beta}{\alpha}\right)}_{\geq 2} \qquad \ldots(\text{using part (a) twice})\\ &\geq 2+2+2\\ &= 6\end{aligned}$$

**Question 13 (c)**

$$v\frac{dv}{dx} = -\left(g+kv^2\right)$$

$$\int_0^H dx = -\int_u^0 \frac{v}{g+kv^2}\,dv \qquad \ldots\left(\text{where } u=\frac{1}{2}\sqrt{\frac{g}{k}}\right)$$

$$= \int_0^u \frac{v}{g+kv^2}\,dv$$

$$H = \frac{1}{2k}\left[\ln\left(g+kv^2\right)\right]_0^u$$

$$= \frac{1}{2k}\ln\left(\frac{g+ku^2}{g}\right)$$

$$= \frac{1}{2k}\ln\left(\frac{g+k\times\frac{1}{4}\times\frac{g}{k}}{g}\right) \qquad \ldots\left(\text{sub in } u=\frac{1}{2}\sqrt{\frac{g}{k}}\right)$$

$$= \frac{1}{2k}\ln\left(\frac{5}{4}\right)$$

**Question 13 (d)**

Let $Q$ have coordinates $Q(x,y)$. The thickness of the arbitrary slice is $\delta x$. If we let $(x,h)$ be the 'coordinate' that maps the height against $x$, then we have the two points $(0,1)$ and $(4,2)$, which make the equation

$$h-1 = \frac{1}{4}(x-0)$$

$$h = \frac{x}{4}+1$$

Hence, the volume of the arbitrary slice is

$$\delta V = \frac{1}{2}\times PQ\times \text{Height}\times\delta x$$

$$= \frac{1}{2}\times 2y\times\left(\frac{x}{4}+1\right)\times\delta x$$

$$= y\left(\frac{x}{4}+1\right)\delta x$$

We now need to find an expression for $y$ in terms of $x$. This is simply the equation of the line $BC$ if you look at the solid from a bird's-eye view.

We are given that the *y* intercept of *BC* is 1 and we can find that $m_{BC} = \dfrac{3-1}{4-0} = \dfrac{1}{2}$.

Hence, *BC* has equation $y = \dfrac{x}{2} + 1$, and now we can find the volume.

$$\begin{aligned} V &= \int_0^4 \left(\frac{x}{2}+1\right)\left(\frac{x}{4}+1\right) dx \\ &= \frac{38}{3} \end{aligned}$$

**Question 13 (e)**

Note that $\overrightarrow{AC}$ is longer than $\overrightarrow{AD}$ by a factor of $\sqrt{2}$ since it is the hypotenuse of a right-angled isosceles triangle. Hence, by rotating vectors, we have

$$\begin{aligned} \overrightarrow{AD} \times \sqrt{2}\,\text{cis}\left(\frac{\pi}{4}\right) &= \overrightarrow{AC} \\ (d-a)(1+i) &= c-a \\ c &= (d-a)(1+i)+a \\ &= d(1+i) - a(1+i) + a \\ &= d(1+i) - ia \end{aligned}$$

**Question 14 (a) (i)**

Cross multiply and equate numerators.

$$16 \equiv (A+2x)(x^2-2x+2)+(B-2x)(x^2+2x+2)$$

Equate the constant term.

$$2A+2B=16$$
$$A+B=8$$

By symmetry, $A$ and $B$ must both be 4.

**Question 14 (a) (ii)**

$$\begin{aligned}
\int_0^m \frac{dx}{x^4+16} &= \int_0^m \frac{4+2x}{x^2+2x+2}+\frac{4-2x}{x^2-2x+2}\,dx \\
&= \int_0^m \frac{2+2x}{x^2+2x+2}+\frac{2}{x^2+2x+2}\,dx-\int_0^m \frac{2x-4}{x^2-2x+2}\,dx \\
&= \int_0^m \frac{2+2x}{x^2+2x+2}+\frac{2}{1+(x+1)^2}\,dx-\int_0^m \frac{2x-2}{x^2-2x+2}-\frac{2}{1+(x-1)^2}\,dx \\
&= \Big[\ln\left(x^2+2x+2\right)+2\tan^{-1}(x+1)\Big]_0^m-\Big[\ln\left(x^2-2x+2\right)-2\tan^{-1}(x-1)\Big]_0^m \\
&= \ln\left(\frac{m^2+2m+2}{m^2-2m+2}\right)+2\tan^{-1}(m+1)+2\tan^{-1}(m-1)+\cancel{\ln 2-\ln 2} \\
&= \ln\left(\frac{m^2+2m+2}{m^2-2m+2}\right)+2\tan^{-1}(m+1)+2\tan^{-1}(m-1)
\end{aligned}$$

**Question 14 (a) (iii)**

Each inverse tan term goes to $2\times\frac{\pi}{2}=\pi$ and the inside of the log function approaches 1.

Hence, the entire log function approaches zero. Therefore, the limit is $2\pi$.

**Question 14 (b) (i)**

They are both angles subtended from the same arc $ED$ in circle $\mathcal{C}_1$.

**Question 14 (b) (ii)**

We know that $\angle EDA = \angle EBA$ for the same reason as (i), but with arc *EA*. But now, looking at circle $\mathcal{C}_2$, we notice that arc *AC* subtends both $\angle ABC$ and $\angle AFC$.

Hence, $\angle EDA = \angle EBA = \angle CBA = \angle AFC$.

**Question 14 (b) (iii)**

First, let $\angle EAD = \theta$, so we already know that $\angle EBD = \theta$ too, from part (i).

From (ii), we can deduce that *AFDG* is a cyclic quadrilateral (angles subtended by the same arc are equal). Hence, $\angle FAD = \angle FGD$ (angles subtended by the same arc are equal).

But $\angle FAD = \pi - \theta$ (angles on the straight line *EF*) so therefore $\angle FGD = \pi - \theta$. Therefore, we have $\angle CGD + \angle CBD = \pi - \theta + \theta = \pi$, which shows that *BCGD* is a cyclic quadrilateral (opposite angles are supplementary).

**Question 14 (c) (i)**

Resolving Horizontally: $R\cos\theta = mrw^2$

Resolving Vertically: $R\sin\theta = mg$

Divide them to get $\tan\theta = \dfrac{g}{rw^2}$. But from the diagram, $\tan\theta = \dfrac{r}{h}$, so equate them.

$$\frac{r}{h} = \frac{g}{rw^2}$$
$$w^2 = \frac{gh}{r^2}$$

**Question 14 (c) (ii)**

Resolving Horizontally: $T\sin\theta - N\cos\theta = mrw^2 \quad ...(1)$

Resolving Vertically: $T\cos\theta + N\sin\theta = mg \quad ...(2)$

Using elimination method $(2)\times\sin\theta + (1)\times\cos\theta$:

$$\begin{aligned} N &= mg\sin\theta - mrw^2\cos\theta \\ &= mg\left(\sin\theta - \frac{rw^2}{g}\cos\theta\right) \\ &= mg\left(\sin\theta - \frac{h}{r}\cos\theta\right) \quad ...(\text{from part (i)}) \end{aligned}$$

**Question 14 (c) (iii)**

We want $N \geq 0$ for the system to be physically feasible.

$$\begin{aligned} mg\left(\sin\theta - \frac{h}{r}\cos\theta\right) &\geq 0 \\ \tan\theta &\geq \frac{h}{r} \\ &\geq \frac{1}{\tan\theta} \quad ...\left(\text{since } \tan\theta = \frac{r}{h}\right) \\ \tan^2\theta &\geq 1 \\ \tan\theta &\geq 1 \\ \theta &\geq \frac{\pi}{4} \end{aligned}$$

**Question 15 (a) (i)**

$$\begin{aligned} I_1 &= \int_0^1 x\sqrt{1-x^2}\,dx \\ &= -\frac{1}{3}\left[\left(1-x^2\right)^{\frac{3}{2}}\right]_0^1 \\ &= \frac{1}{3} \end{aligned}$$

**Question 15 (a) (ii)**

We make the following choice for $u$ and $dv$

$$\begin{aligned} u &= x^{n-1} & dv &= x\sqrt{1-x^2}\,dx \\ du &= (n-1)x^{n-2} & v &= -\frac{1}{3}\left(1-x^2\right)^{\frac{3}{2}} \end{aligned}$$

Now, use integration by parts

$$\begin{aligned} I_n &= \cancel{\left[-\frac{1}{3}x^{n-1}\left(1-x^2\right)^{\frac{3}{2}}\right]_0^1} + \frac{n-1}{3}\int_0^1 x^{n-2}\left(1-x^2\right)^{\frac{3}{2}}\,dx \\ &= \frac{n-1}{3}\int_0^1 x^{n-2}\left(1-x^2\right)\sqrt{1-x^2}\,dx \\ &= \frac{n-1}{3}\int_0^1 \left(x^{n-2}-x^n\right)\sqrt{1-x^2}\,dx \\ &= \frac{n-1}{3}\int_0^1 x^{n-2}\sqrt{1-x^2}-x^n\sqrt{1-x^2}\,dx \\ &= \frac{n-1}{3}\left(I_{n-2}-I_n\right) \end{aligned}$$

Re-arrange to make $I_n$ the subject.

$$\begin{aligned} 3I_n &= (n-1)I_{n-2} - (n-1)I_n \\ (n+2)I_n &= (n-1)I_{n-2} \\ I_n &= \frac{n-1}{n+2}I_{n-2} \end{aligned}$$

**Question 15 (a) (iii)**

From part (i), we know $I_1 = \frac{1}{3}$. Use the recurrence to get

$$\begin{aligned} I_3 &= \frac{2}{5} I_1 \\ &= \frac{2}{15} \end{aligned}$$

and once more to get

$$\begin{aligned} I_5 &= \frac{4}{7} I_3 \\ &= \frac{8}{105} \end{aligned}$$

**Question 15 (b) (i)**

Using implicit differentiation, we have

$$\begin{aligned} \frac{1}{2\sqrt{x}} + \frac{y'}{2\sqrt{y}} &= 0 \\ y' &= -\sqrt{\frac{y}{x}} \end{aligned}$$

Hence, the gradient of the tangent at $P$ is $m = -\sqrt{\frac{d}{c}}$.

Use the point-gradient formula to find the equation of the tangent.

$$\begin{aligned} y - d &= -\sqrt{\frac{d}{c}}(x - c) \\ y\sqrt{c} - d\sqrt{c} &= -x\sqrt{d} + c\sqrt{d} \\ x\sqrt{d} + y\sqrt{c} &= d\sqrt{c} + c\sqrt{d} \end{aligned}$$

**Question 15 (b) (ii)**

The $x$ intercept of the tangent is

$$\begin{aligned} OA &= \frac{d\sqrt{c}+c\sqrt{d}}{\sqrt{d}} \\ &= \sqrt{cd}+c \end{aligned}$$

Similarly, the $y$ intercept is $OB = \sqrt{cd}+d$. Now we can add them.

$$\begin{aligned} OA+OB &= 2\sqrt{cd}+c+d \\ &= \left(\sqrt{c}+\sqrt{d}\right)^2 \\ &= \left(\sqrt{a}\right)^2 \qquad \ldots\left(\text{since } P \text{ lies on } \sqrt{x}+\sqrt{y}=\sqrt{a}\right) \\ &= a \end{aligned}$$

**Question 15 (c) (i)**

$P$ lies on both the ellipse and hyperbola. Hence, we have two equations.

$$\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}=1$$
$$\frac{x_1^2}{c^2}-\frac{y_1^2}{d^2}=1$$

Multiplying both by $a^2b^2$ and equating, we have

$$\begin{aligned} \frac{x_1^2}{a^2}+\frac{y_1^2}{b^2} &= \frac{x_1^2}{c^2}-\frac{y_1^2}{d^2} \\ x_1^2\left(\frac{1}{a^2}-\frac{1}{c^2}\right) &= -y_1^2\left(\frac{1}{b^2}+\frac{1}{d^2}\right) \\ x_1^2\left(\frac{c^2-a^2}{a^2c^2}\right) &= -y_1^2\left(\frac{b^2+d^2}{b^2d^2}\right) \\ \frac{x_1^2}{y_1^2} &= \frac{a^2c^2}{\left(c^2-a^2\right)}\times-\frac{\left(b^2+d^2\right)}{b^2d^2} \\ &= \frac{a^2c^2}{\left(a^2-c^2\right)}\times\frac{\left(b^2+d^2\right)}{b^2d^2} \end{aligned}$$

**Question 15 (c) (ii)**

The tangent has the following gradient at $P$

$$\frac{2x_1}{a^2}+\frac{2y_1y'}{b^2}=0$$

$$y'=-\frac{b^2x_1}{a^2y_1}$$

Similarly, the hyperbola has gradient $y'=\frac{d^2x_1}{c^2y_1}$.

Multiply their gradient to get $m_1m_2=-\frac{b^2d^2}{a^2c^2}\times\frac{x_1^{\,2}}{y_1^{\,2}}$.

If the two conics have the same foci, then $ae_E=ce_H$ and hence $a^2-b^2=c^2+d^2$, since $a^2e^2=a^2-b^2$ in an ellipse and $c^2e^2=c^2+d^2$ in the hyperbola.

Substitute into (i)

$$\begin{aligned}m_1m_2&=-\frac{b^2d^2}{a^2c^2}\times\frac{a^2c^2}{\left(a^2-c^2\right)}\times\frac{\left(b^2+d^2\right)}{b^2d^2}\\&=-\frac{b^2+d^2}{a^2-c^2}\qquad\ldots\left(\text{since } a^2-b^2=c^2+d^2\right)\\&=-1\end{aligned}$$

**Question 16 (a) (i)**

Standard result.

$$\begin{aligned}\alpha^k + \alpha^{-k} &= \text{cis}\left(k\theta\right) + \text{cis}\left(-k\theta\right) \\ &= 2\,\text{Re}\left(\text{cis}\left(k\theta\right)\right) \\ &= 2\cos\left(k\theta\right)\end{aligned}$$

**Question 16 (a) (ii)**

The series is a sum of a GP. Note that $\dfrac{1}{\alpha} = \text{cis}\left(-\theta\right) = \bar{\alpha}$.

$$\begin{aligned}C &= \frac{\alpha^{-n}\left(1-\alpha^{2n+1}\right)}{1-\alpha} \\ &= \frac{\alpha^{-n}\left(1-\alpha^{2n+1}\right)\left(1-\alpha^{-1}\right)}{\left(1-\alpha\right)\left(1-\bar{\alpha}\right)} \\ &= \frac{\left(\alpha^{-n}-\alpha^{n+1}\right)\left(1-\alpha^{-1}\right)}{\left(1-\alpha\right)\left(1-\bar{\alpha}\right)} \\ &= \frac{\alpha^{-n}-\alpha^{-n-1}-\alpha^{n+1}+\alpha^{n}}{\left(1-\alpha\right)\left(1-\bar{\alpha}\right)} \\ &= \frac{\alpha^{n}+\alpha^{-n}-\left(\alpha^{n+1}+\alpha^{-(n+1)}\right)}{\left(1-\alpha\right)\left(1-\bar{\alpha}\right)}\end{aligned}$$

**Question 16 (a) (iii)**

Substitute in (i) repeatedly to the series definition of $C$.

$$\begin{aligned} C &= 1+\left(\alpha+\alpha^{-1}\right)+\left(\alpha^2+\alpha^{-2}\right)+\ldots+\left(\alpha^n+\alpha^{-n}\right) \\ &= 1+2\cos\theta+2\cos 2\theta+\ldots+2\cos n\theta \end{aligned}$$

Now substitute into the closed form for $C$ in (ii).

$$\begin{aligned} C &= \frac{\left(\alpha^n+\alpha^{-n}\right)-\left(\alpha^{n+1}+\alpha^{-(n+1)}\right)}{\left(1-\alpha\right)\left(1-\bar{\alpha}\right)} \\ &= \frac{2\cos n\theta-2\cos\left(n+1\right)\theta}{1-\left(\alpha+\bar{\alpha}\right)+\alpha\bar{\alpha}} \\ &= \frac{2\cos n\theta-2\cos\left(n+1\right)\theta}{1-\left(2\operatorname{Re}\alpha\right)+1} \\ &= \frac{2\cos n\theta-2\cos\left(n+1\right)\theta}{2-2\cos\theta} \\ &= \frac{\cos n\theta-\cos\left(n+1\right)\theta}{1-\cos\theta} \end{aligned}$$

Equate the two expressions and we acquire the result.

**Question 16 (a) (iv)**

Re-arrange the result in (iii) to get

$$1+2\cos\theta+2\cos 2\theta+\ldots+2\cos n\theta=\frac{\cos n\theta-\cos(n+1)\theta}{1-\cos\theta}$$

$$\cos\theta+\cos 2\theta+\ldots+\cos n\theta=\frac{1}{2}\left(\frac{\cos n\theta-\cos(n+1)\theta}{1-\cos\theta}-1\right)$$

Let $\theta=\dfrac{\pi}{n}$.

$$\begin{aligned}\cos\left(\frac{\pi}{n}\right)+\cos\left(\frac{2\pi}{n}\right)+\ldots+\cos\left(\frac{n\pi}{n}\right)&=\frac{1}{2}\left(\frac{\cos\left(\dfrac{n\pi}{n}\right)-\cos\left(\dfrac{n+1}{n}\pi\right)}{1-\cos\left(\dfrac{\pi}{n}\right)}-1\right)\\&=\frac{1}{2}\left(\frac{-1-\cos\left(\pi+\dfrac{\pi}{n}\right)}{1-\cos\left(\dfrac{\pi}{n}\right)}-1\right)\\&=\frac{1}{2}\left(\frac{-1+\cos\left(\dfrac{\pi}{n}\right)}{1-\cos\left(\dfrac{\pi}{n}\right)}-1\right)\\&=-1\end{aligned}$$

And this is independent of *n*.

**Question 16 (b)**

We are given that $e=2$ and $|a|e\pm|a|=1$, so solving for $a$ yields

$$\begin{aligned}|a|e\pm|a|&=1\\|a|(e\pm1)&=1\\a&=\pm\frac{1}{3}\text{ or }a=\pm1\end{aligned}$$

**Question 16 (c) (i)**

Case #1: Tile *C* is the same colour as *B*.

| | |
|---|---|
| A | B |
| B | ? |

Tile *D* can be anything except the colour of *B*, so therefore there are $(x-1)$ possibilities.

Case #2: Tile *C* is a different colour to *B*.

| | |
|---|---|
| A | ? |
| B | ? |

Tile *C* can be anything except *A* or *B*'s colour, so it has $(x-2)$ possibilities. Similarly, tile *D* can be anything except *B* or *C*'s colour, so it also has $(x-2)$ possibilities.

Hence, the total is $(x-2)^2+(x-1)=x^2-3x+3$ possibilities.

**Question 16 (c) (ii)**

Base Case: $n=1$

We have a $2\times1$ grid, which has $x(x-1)$ possibilities (given), which matches the formula for $n=1$.

Inductive Hypothesis: $n=k$

A $2\times k$ grid will have $x(x-1)\left(x^2-3x+3\right)^{k-1}$ possible colourings.

Inductive Step: $n=k\Rightarrow n=k+1$

Consider a $2\times(k+1)$ grid, which is basically a slight extension of the grid in the inductive hypothesis. Use the result from (i) here.

Before the extension, we had $x(x-1)\left(x^2-3x+3\right)^{k-1}$ possible colourings. By throwing on the extra two tiles, we multiply in the result by the number of possibilities in (i), which is $\left(x^2-3x+3\right)$. Hence, we have

$$x(x-1)\left(x^2-3x+3\right)^{k-1}\times\left(x^2-3x+3\right)=x(x-1)\left(x^2-3x+3\right)^{k}$$

and the induction is complete.

**Question 16 (c) (iii)**

Substitute $x=3$ into (ii) to get 486. But this will include a number of 'bad' cases because the equation in (ii) was constructed under the assumption that not all colours have to be used.

However, in for this question, we must use all colours at least once. So we will manually subtract the 'bad' case where we use only two colours (note that we can't use one colour only).

Choose 3 out of 2 colours to be used in $\binom{3}{2}$ ways. But once we pick the two colours, we have only two possible colourings that zig-zag across the grid in an alternating manner.

Hence, there are $2\times\binom{3}{2}=6$ bad cases.

Therefore, the total number of colourings is $486-6=480$ possible colourings.