2016 HSC Examination

Mathematics Extension 2

SOLUTIONS

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Multiple Choice

1. C
2. C
3. A
4. D
5. B
6. C
7. D
8. C
9. A
10. B

Brief Explanations

Question 1 The graph is symmetric in the *y* axis only, so it must be (C). Option (A) lies in the first quadrant only. Option (B) and (D) imply that *y* can be negative.

Question 2 Only (C) satisfies the condition $f(1) = f'(1) = 0$.

Question 3 The circle has $e = 0$ and the ellipse has $0 < e < 1$, so it must be (A). The parabola has $e = 1$, which eliminates (B) and (C). The hyperbola has $e > 1$, which eliminates (D).

Question 4 Note that $w - z = \overrightarrow{ZW}$. Connect the base of $\overrightarrow{ZW}$ to the origin. Options (A) and (C) lie in the fourth quadrant. Option (B) lies in the second quadrant.

Question 5 Recall that $\arg\left(\frac{1-i}{1+i}\right) = \arg(1-i) - \arg(1+i) = -\frac{\pi}{4} - \frac{\pi}{4} = -\frac{\pi}{2}$. Since this is negative, this is a clock-wise rotation.

Question 6 Note that $P(x+1) = 1 + (x+1) + ... + (x+1)^{12}$. To get the coefficient of x^8, we extract the coefficient from each of the terms $(1+x)^k$ for $k = 8, ..., 12$, which respectively yields $\binom{8}{k}$ for $k = 8, ..., 12$. Add these up.

Question 7 Recall that the curve $x^2 - y^2 = k$ rotated 45° anti-clockwise is $xy = \frac{k}{2}$.

Question 8 Standard resolving forces question.

Question 9 Let the front-facing side of the rectangular cross section be *y*, and the other side be *x*. We have the pairs $(h, y) = (0, 2)$ and $(h, y) = (4, 8)$. Hence the gradient of the linear relationship between *y* and *h* is $m = \frac{8-2}{4-0} = \frac{3}{2}$. Thus, the relationship is $y = \frac{3h}{2} + 2$.

Use the same technique to find *x* in terms of *h*. The volume is given by $\int_0^4 xy\,dh$.

Question 10 $x + \frac{1}{x} = -1$ is equivalent to $x^2 + x + 1 = 0$, implying that *x* is a complex cube root of 1, except with *x* instead of the usual ω. Hence $x^3 = 1$. Since 2016 is a multiple of 3, we have $x^{2016} = \frac{1}{x^{2016}} = 1$ and hence the answer is (B).

Written Response

Question 11 (a) (i)

We have $|z| = \sqrt{3+1} = 2$ and $\tan\theta = -\frac{1}{\sqrt{3}}$. Since z is in the fourth quadrant, $\theta = -\frac{\pi}{6}$.

Hence $z = 2\text{cis}\left(-\frac{\pi}{6}\right)$.

Question 11 (a) (ii)

Use De Moivre's Theorem

$$\begin{aligned} z^6 &= 2^6 \text{cis}\left(-\frac{\pi}{6}\times 6\right) \\ &= 64\text{cis}\left(-\pi\right) \\ &= -64 \end{aligned}$$

Question 11 (a) (iii)

Adding a power of z rotates the point 30° clockwise about the origin. Since the argument is already -30°, we need to only apply it two more times. Hence $n = 3$.

Check: $z^3 = 8\text{cis}\left(-\frac{\pi}{2}\right) = -8i$, which is purely imaginary.

Question 11 (b)

Use integration by parts.

$$\begin{aligned} u &= x & dv &= e^{-2x} \\ du &= dx & v &= -\frac{1}{2}e^{-2x}dx \end{aligned}$$

Hence

$$\begin{aligned} I &= -\frac{1}{2}xe^{-2x} + \frac{1}{2}\int e^{-2x}dx \\ &= -\frac{1}{2}xe^{-2x} - \frac{1}{4}e^{-2x} + C \end{aligned}$$

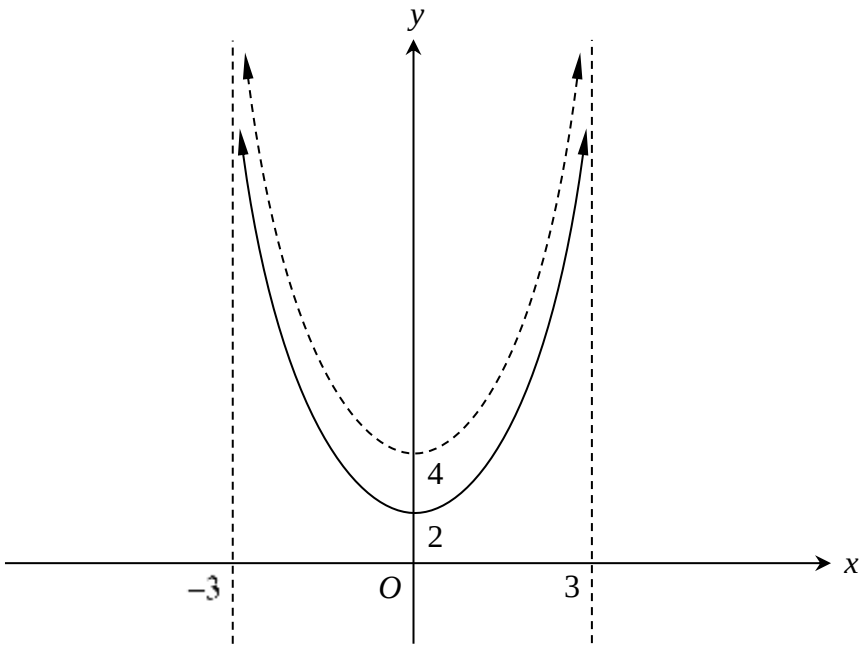
Question 11 (c)

Use implicit differentiation, noting that differentiating the term in the right hand side will require the use of the product rule.

$$\begin{aligned} 3x^2 + 3y^2 y' &= 2\left(y + xy'\right) \\ 3x^2 + 3y^2 y' &= 2y + 2xy' \\ y' &= \frac{2y - 3x^2}{3y^2 - 2x} \end{aligned}$$

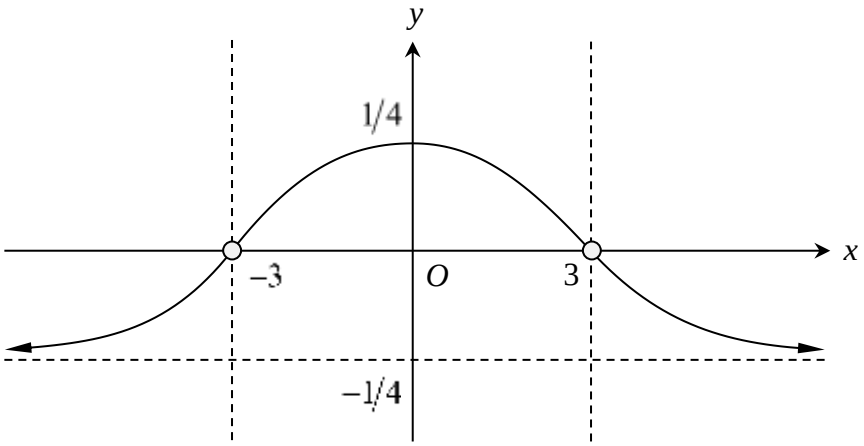
Question 11 (d) (i)

The parts of the original curve that were below the x axis are not mapped to anything in the real plane, so they 'disappear'.



Question 11 (d) (ii)

The vertical asymptotes map to '*x* intercepts', except they are open circles because the point is not actually defined. Also, there is not enough information to guarantee that the curve is actually smooth around $x = \pm 3$, but it was drawn as such for purely aesthetic reasons.



Question 11 (e)

The domain comes purely from $\sin^{-1}\left(\frac{x}{2}\right)$ because the *x* in front does not affect the domain.

So the domain is simply $-2 \le x \le 2$.

For the range, note that the function is even since it is the product of two odd functions. Hence, it suffices to consider only the right half. Since $y = x$ and $y = \sin^{-1}\left(\frac{x}{2}\right)$ are monotonically increasing in the interval $0 \le x \le 2$, the product is also monotonically increasing in same interval. So the lowest extremity of the range is at $f(0) = 0$ and the other extremity of the range is at $f(2) = \pi$. Hence the range is $0 \le y \le \pi$.

Question 12 (a) (i)

From the diagram, we can see that $a = 3$ and $b = 2$. Hence the equation is $\frac{x^2}{9} + \frac{y^2}{4} = 1$.

Question 12 (a) (ii)

Recall that $b^2 = a^2\left(1 - e^2\right)$. Therefore $e^2 = \frac{a^2 - b^2}{a^2} = \frac{9-4}{9} = \frac{5}{3}$ and hence $e = \frac{\sqrt{5}}{3}$.

Question 12 (a) (iii)

From $b^2 = a^2\left(1-e^2\right)$, we have $a^2e^2 = a^2 - b^2 = 5$. Hence $ae = \sqrt{5}$, and thus $S\left(\pm\sqrt{5}, 0\right)$.

Question 12 (a) (iv)

The directrices are $x = \pm\dfrac{a}{e} = \pm\dfrac{3}{\sqrt{5}/3} = \pm\dfrac{9}{\sqrt{5}}$.

Question 12 (b) (i)

Let $y = x\,f\left(x\right) - \displaystyle\int x\,f'\left(x\right)dx$. To differentiate $x\,f\left(x\right)$, we use the product rule. To differentiate the integral, we 'cancel' out the integration process, leaving behind the integrand.

$$\frac{dy}{dx} = f\left(x\right) + x\,f'\left(x\right) - x\,f'\left(x\right) = f\left(x\right)$$

In other words, the primitive of $f\left(x\right)$ is $F\left(x\right) = x\,f\left(x\right) - \displaystyle\int x\,f'\left(x\right)dx$.

Question 12 (b) (ii)

Let $f\left(x\right) = \tan^{-1}\left(x\right)$.

Using (i), the primitive is

$$\begin{aligned} F\left(x\right) &= x\tan^{-1}\left(x\right) - \int \frac{x}{1+x^2}\,dx \\ &= x\tan^{-1}\left(x\right) - \frac{1}{2}\ln\left(1+x^2\right) + C \end{aligned}$$

Question 12 (c) (i)

Let $c = \cos\theta$ and $s = \sin\theta$.

Using De Moivre's Theorem, we have $z^4 = \cos 4\theta + i\sin 4\theta$.

Expanding $z^4 = \left(c+is\right)^4$ explicitly, we have $z^4 = \mathrm{c}^4 + 4ic^3s - 6c^2s^2 - 4ics^3 + s^4$.

Equate the real terms to get $\cos 4\theta = c^4 - 6c^2s^2 + s^4$, as required.

Question 12 (c) (ii)

Continuing on from our result in (i)

$$\begin{aligned}\cos 4\theta &= c^4 - 6c^2s^2 + s^4 \\ &= c^4 - 6c^2\left(1-c^2\right) + \left(1-c^2\right)^2 \\ &= c^4 - 6c^2 + 6c^4 + 1 - 2c^2 + c^4 \\ &= 8c^4 - 8c^2 + 1\end{aligned}$$

Question 12 (d) (i)

Using parametric differentiation, we have

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy/dp}{dx/dp} \\ &= \frac{-c/p^2}{c} \\ &= -\frac{1}{p^2}\end{aligned}$$

Hence, the gradient of the normal is p^2.

Use the point-gradient formula.

$$\begin{aligned}y - \frac{c}{p} &= p^2\left(x - cp\right) \\ &= p^2x - cp^3\end{aligned}$$

Divide both sides by p.

$$\begin{aligned}\frac{y}{p} - \frac{c}{p^2} &= px - cp^2 \\ px - \frac{y}{p} &= cp^2 - \frac{c}{p^2} \\ &= c\left(p^2 - \frac{1}{p^2}\right)\end{aligned}$$

Question 12 (d) (ii)

Substitute Q into the equation in (i)

$$\begin{aligned} cpq - \frac{c}{pq} &= c\left(p^2 - \frac{1}{p^2} \right) \\ pq - \frac{1}{pq} &= p^2 - \frac{1}{p^2} \\ p^2 - pq &= \frac{1}{p^2} - \frac{1}{pq} \\ &= \frac{pq - p^2}{p^3 q} \\ -1 &= \frac{1}{p^3 q} \\ q &= -\frac{1}{p^3} \end{aligned}$$

Question 13 (a)

$$\begin{aligned} \frac{d}{dx}\left[\ln\left(f\left(x \right) \right) \right] &= \frac{f'\left(x \right)}{f\left(x \right)} \\ \frac{d}{dx}\left[\ln\left(x^x \right) \right] &= \frac{f'\left(x \right)}{x^x} \\ \frac{d}{dx}\left[x \ln x \right] &= \frac{f'\left(x \right)}{x^x} \\ f'\left(x \right) &= x^x \times \frac{d}{dx}\left[x \ln x \right] \\ &= x^x \left(1 + \ln x \right) \\ &= 0 \quad \text{for stationary points} \end{aligned}$$

Since $x > 0$, we have $x^x \neq 0$ and hence $1 + \ln x = 0$, which yields $x = \frac{1}{e}$.

Clearly if $0 < x < \frac{1}{e}$, then $f'\left(x \right) = x^x \left(1 + \ln x \right) < 0$ since $x^x > 0$ and $1 + \ln x < 0$. Similarly, if $x > \frac{1}{e}$, then $x^x > 0$ and $1 + \ln x > 0$, hence $f'\left(x \right) > 0$. These results combined imply that the stationary point at $x = \frac{1}{e}$ is a *minimum* stationary point. Alternatively, one may just use the standard table of values and test points.

Question 13 (b)

Let $\angle BCP = \theta$. It suffices to show that $\angle OQC = \theta$ as that would imply that ΔOQP is isosceles, and hence the result $OP = OQ$.

Observe that $OCQA$ is a cyclic quadrilateral since $\angle OAQ = \angle OCP = \dfrac{\pi}{2}$ (tangent $\perp$ radius at the point of contact).

$$\begin{aligned} \angle OQC &= \angle OAC && \ldots(\text{angles subtended by common arc } OC) \\ &= \angle BCP && \ldots(\text{alternate segment theorem}) \\ &= \theta \end{aligned}$$

As required.

Question 13 (c) (i)

To find the semi-vertical angle, we use the tangent ratio to get $\tan(\angle BAC) = \tan\theta = \dfrac{3}{4}$.

Resolve vertically:

$$\begin{aligned} T_1 \cos\theta &= Mg \\ T_1\left(\frac{4}{5}\right) &= 10M \qquad \ldots\left(\text{since } \tan\theta = \frac{3}{4}\right) \\ T_1 &= \frac{25M}{2} \end{aligned}$$

Resolve horizontally:

$$\begin{aligned} T_1 \sin\theta + T_2 &= M(0.3)\omega^2 \\ \frac{25M}{2}\left(\frac{3}{5}\right) + T_2 &= M(0.3)\omega^2 \\ \frac{15M}{2} + T_2 &= 0.3M\omega^2 \\ T_2 &= 0.3M\omega^2 - \frac{15M}{2} \\ &= 0.3M\left(\omega^2 - 25\right) \end{aligned}$$

Question 13 (c) (ii)

Let $T_2 > T_1$

$$\begin{aligned} 0.3M\left(\omega^2 - 25\right) &> \frac{25M}{2} \\ \omega^2 - 25 &> \frac{125}{3} \\ \omega^2 &> \frac{200}{3} \\ \omega &> \sqrt{\frac{200}{3}} \quad \ldots.(\omega > 0) \end{aligned}$$

Question 13 (d) (i)

Consider $p'(x) = 3ax^2 + 2bx + c$ and the discriminant

$$\begin{aligned} \Delta &= 4b^2 - 12ac \\ &= 4\left(b^2 - 3ac\right) \\ &< 0 \quad \ldots(\text{since } b^2 - 3ac < 0) \end{aligned}$$

Hence, $p'(x)$ is never zero. In other words, it is always positive or always negative, which respectively implies that $p(x)$ is monotonically increasing or decreasing. In either case, this will result in the cubic having only one real root.

Question 13 (d) (ii)

We are given that $\Delta = 0$, so $p'(x)$ has a double root. Solving $p'(x) = 0$ explicitly, we have

$$\begin{aligned} x &= \frac{-2b \pm 2\sqrt{b^2 - 3ac}}{6a} \\ &= -\frac{b}{3a} \qquad \ldots(\text{since } \Delta = 0) \end{aligned}$$

Hence $p'\left(-\frac{b}{3a}\right) = 0$. But since this is the double root, then $p''\left(-\frac{b}{3a}\right) = 0$ as well, by the multiple root theorem. So $p\left(-\frac{b}{3a}\right) = p'\left(-\frac{b}{3a}\right) = p''\left(-\frac{b}{3a}\right) = 0$ and hence it is a root of multiplicity 3 (triple root).

Question 14 (a) (i)

We can just differentiate the right hand side instead of actually computing the integral.

$$\begin{aligned}\frac{d}{dx}\left(\frac{1}{3}\cos^3 x-\cos x+C\right) &= \cos^2 x\times(-\sin x)+\sin x \\ &= \sin x\left(1-\cos^2 x\right) \\ &= \sin^3 x\end{aligned}$$

Question 14 (a) (ii)

Let $u = x-\frac{\pi}{2}$, so $du = dx$.

$$\begin{aligned}I &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^{2n-1}\left(u+\frac{\pi}{2}\right)du \\ &= (-1)^{2n-1}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^{2n-1}(u)\,du \\ &= 0 \qquad \ldots(\text{since the integrand is an odd function})\end{aligned}$$

Question 14 (a) (iii)

The radius of the shell is x, the height is y and the thickness is δx. By unfolding the shell into a thin rectangular prism, the volume of the shell is $\delta V = 2\pi xy\,\delta x = 2\pi x\sin^3 x\,\delta x$.

Using integration by parts and integral already found in (i), we have

$$\begin{aligned}V &= 2\pi\int_0^{\pi} x\sin^3 x\,dx \\ &= 2\pi\left[\frac{x}{3}\cos^3 x - x\cos x\right]_0^{\pi} - \underbrace{\int_0^{\pi}\frac{1}{3}\cos^3 x-\cos x\,dx}_{\substack{=0 \\ \text{using the result in (ii)}}} \\ &= 2\pi\left[-\frac{\pi}{3}+\pi\right] \\ &= \frac{4\pi^2}{3}\end{aligned}$$

Question 14 (b) (i)

$$\begin{aligned}
I_0 &= \int_0^1 \frac{dx}{\left(x^2+1\right)^2} \\
&= \int_0^{\frac{\pi}{4}} \frac{\sec^2\theta}{\left(\tan^2\theta+1\right)^2}\,d\theta \qquad \ldots\left(\text{using } u=\tan\theta\right) \\
&= \int_0^{\frac{\pi}{4}} \frac{d\theta}{\sec^2\theta} \\
&= \int_0^{\frac{\pi}{4}} \cos^2\theta\,d\theta \\
&= \frac{1}{2}\int_0^{\frac{\pi}{4}} 1+\cos 2\theta\,d\theta \\
&= \frac{1}{2}\left[\theta+\frac{1}{2}\sin 2\theta\right]_0^{\frac{\pi}{4}} \\
&= \frac{1}{2}\left[\frac{\pi}{4}+\frac{1}{2}\right] \\
&= \frac{\pi}{8}+\frac{1}{4}
\end{aligned}$$

Question 14 (b) (ii)

Combine the integrals.

$$\begin{aligned}
I_0+I_2 &= \int_0^1 \frac{1+x^2}{\left(x^2+1\right)^2}\,dx \\
&= \int_0^1 \frac{dx}{x^2+1} \\
&= \tan^{-1}1-\tan^{-1}0 \\
&= \frac{\pi}{4}
\end{aligned}$$

Question 14 (b) (iii)

We use the "plus one, minus one" trick to reduce the integral.

$$\begin{aligned}
I_4 &= \int_0^1 \frac{x^4}{\left(x^2+1\right)^2}\,dx \\
&= \int_0^1 \frac{1-\left(1-x^4\right)}{\left(x^2+1\right)^2}\,dx \\
&= \int_0^1 \frac{1}{\left(x^2+1\right)^2} - \frac{1-x^2}{1+x^2}\,dx \\
&= I_0 + \int_0^1 \frac{x^2-1}{x^2+1}\,dx \\
&= I_0 + \int_0^1 1 - \frac{2}{1+x^2}\,dx \\
&= I_0 + \left[x - 2\tan^{-1}x\right]_0^1 \\
&= \frac{\pi}{8} + \frac{1}{4} + 1 - \frac{\pi}{2} \\
&= \frac{5}{4} - \frac{3\pi}{8}
\end{aligned}$$

Question 14 (c)

Consider $f(x) = (x-1)\left(\sqrt{x}-1\right)$.

Case #1: $0 \le x \le 1$

We have both factors $(x-1)$ and $\left(\sqrt{x}-1\right)$ being negative, hence $f(x) \ge 0$.

Case #2: $x \ge 1$

Similarly, both factors are positive and again $f(x) \ge 0$.

$$\begin{aligned}
f(x) &\ge 0 \\
(x-1)\left(\sqrt{x}-1\right) &\ge 0 \\
x\sqrt{x} - x - \sqrt{x} + 1 &\ge 0 \\
x\sqrt{x} + 1 &\ge x + \sqrt{x}
\end{aligned}$$

as required.

Question 15 (a)

Let $t=\alpha_i^2$, where α_i represents any root of $P(x)=x^3-3x+1$.

Since α_i is a root, we have $P(\alpha_i)=0$.

$$\begin{aligned} P(\alpha_i) &= P\left(\sqrt{t}\right) \\ &= t\sqrt{t}-3\sqrt{t}+1 \\ &= 0 \end{aligned}$$

Making this a polynomial in terms of *t*, this becomes

$$\begin{aligned} t\sqrt{t}-3\sqrt{t}+1&=0 \\ \sqrt{t}\left(t-3\right)&=-1 \\ t\left(t-3\right)^2&=1 \\ t^3-6t^2+9t-1&=0 \end{aligned}$$

Hence, the $P(t)=t^3-6t^2+9t-1$ is the cubic equation with roots α^2, β^2 and γ^2.

Question 15 (b) (i)

We have $\dfrac{d}{dx}\left(\dfrac{1}{2}v^2\right)=-\dfrac{\mu^2}{x^2}$, which we can solve explicitly for *v*.

$$\frac{1}{2}v^2=\frac{\mu^2}{x}+C$$

When $x=b$, $v=0$ since the particle is at rest at *B*, so substituting this in yields $C=-\dfrac{\mu^2}{b}$.

$$\begin{aligned} \frac{1}{2}v^2&=\frac{\mu^2}{x}-\frac{\mu^2}{b} \\ v^2&=2\mu^2\left(\frac{b-x}{bx}\right) \\ v&=\pm\mu\sqrt{2}\sqrt{\frac{b-x}{bx}} \end{aligned}$$

The only stationary point is $x=b$, which is where the particle was initially. Since it is given that the particle is moves to the left, it must be the case that $v<0$, and hence we choose the negative case $\dfrac{dx}{dt}=-\mu\sqrt{2}\sqrt{\dfrac{b-x}{bx}}$.

Question 15 (b) (ii)

We solve the differential equation proven in (i).

$$\int_b^d \sqrt{\frac{bx}{b-x}}\,dx = -\mu\sqrt{2}\int_0^T dt$$

We use the given substitution $x = b\cos^2\theta$ to find the new limits for the left integral.

First, for when $x = d$, we have $\theta = \cos^{-1}\left(\sqrt{\frac{d}{b}}\right)$.

When $x = b$, $\theta = 0$

Also, for the differential term

$$\begin{aligned} x &= b\cos^2\theta \\ dx &= -2b\cos\theta\sin\theta\,d\theta \end{aligned}$$

Substituting this all in

$$\begin{aligned} -\mu\sqrt{2}\,T &= \int_b^d \sqrt{\frac{bx}{b-x}}\,dx \\ &= -2b\int_0^{\cos^{-1}\sqrt{\frac{d}{b}}} \sqrt{\frac{b^2\cos^2\theta}{b-b\cos^2\theta}}\cos\theta\sin\theta\,d\theta \\ &= -2b\int_0^{\cos^{-1}\sqrt{\frac{d}{b}}} \sqrt{\frac{b\cos^2\theta}{\sin^2\theta}}\cos\theta\sin\theta\,d\theta \\ &= -2b\sqrt{b}\int_0^{\cos^{-1}\sqrt{\frac{d}{b}}} \cot\theta\cos\theta\sin\theta\,d\theta \\ &= -2b\sqrt{b}\int_0^{\cos^{-1}\sqrt{\frac{d}{b}}} \cos^2\theta\,d\theta \end{aligned}$$

Hence

$$\begin{aligned} -\mu\sqrt{2}\,T &= -2b\sqrt{b}\int_0^{\cos^{-1}\left(\frac{d}{b}\right)} \cos^2\theta\,d\theta \\ T &= \frac{2b\sqrt{b}}{\mu\sqrt{2}}\int_0^{\cos^{-1}\left(\frac{d}{b}\right)} \cos^2\theta\,d\theta \\ &= \frac{b\sqrt{2b}}{\mu}\int_0^{\cos^{-1}\left(\frac{d}{b}\right)} \cos^2\theta\,d\theta \end{aligned}$$

Question 15 (b) (iii)

We are given that

$$T = \frac{1}{\mu}\sqrt{\frac{b}{2}}\left(\sqrt{bd - d^2} + b\cos^{-1}\sqrt{\frac{d}{b}}\right).$$

For the particle to reach *O*, we set $d \to 0$.

$$\begin{aligned} T &\to \frac{1}{\mu}\sqrt{\frac{b}{2}}\left(b\cos^{-1}0\right) \\ &= \frac{b}{\mu}\sqrt{\frac{b}{2}}\left(\frac{\pi}{2}\right) \end{aligned}$$

Question 15 (c) (i)

We are given that

$$\frac{3!}{x(x+1)(x+2)(x+3)} = \frac{a_0}{x} + \frac{a_1}{x+1} + \frac{a_2}{x+2} + \frac{a_3}{x+3}$$

Multiply both sides by *x*

$$\frac{3!}{(x+1)(x+2)(x+3)} = a_0 + \left(\frac{x}{x+1}\right)a_1 + \left(\frac{x}{x+2}\right)a_2 + \left(\frac{x}{x+3}\right)a_3$$

Letting $x = 0$ immediately yields $a_0 = \frac{3!}{3!} = 1$.

Repeating a similar process for $(x+1)$, $(x+2)$ and $(x+3)$, but substituting in $x = -1$, $x = -2$ and $x = -3$ respectively, we then get (also respectively)

$$\begin{aligned} a_1 &= \frac{3!}{(-1)(1)(2)} = -3 \\ a_2 &= \frac{3!}{(-2)(-1)(1)} = 3 \\ a_3 &= \frac{3!}{(-3)(-2)(-1)} = -1 \end{aligned}$$

as required.

Question 15 (c) (ii)

We will do the exact same thing for this question as we did above. To shorten the work, we will use sigma notation.

$$\frac{n!}{x(x+1)...(x+n)} = \sum_{m=0}^{n} \frac{a_m}{x+m}$$

Observe that the denominator is an 'incomplete' factorial. We will 'complete the factorial' and realise that a binomial term can be constructed from this.

$$\frac{n!}{x(x+1)...(x+n)} = \sum_{m=0}^{n} \frac{a_m}{x+m}$$

Multiply both sides by some arbitrary factor $x+k$, where $0 \le k \le n$.

$$\frac{n!}{x...(x+(k-1))(x+(k+1))...(x+n)} = a_k + \sum_{\substack{m=0 \\ m \neq k}}^{n} a_k \left(\frac{x+k}{x+m} \right)$$

Let $x = -k$.

$$\begin{aligned} a_k &= \frac{n!}{(-k)...(-k+(k-1))(-k+(k+1))...(-k+n)} \\ &= \frac{n!}{\underbrace{(-k)...(-1)}_{(-1)^k k!} \underbrace{(1)...(-k+n)}_{(n-k)!}} \\ &= \frac{n!}{(-1)^k k!(n-k)!} \\ &= (-1)^k \binom{n}{k} \end{aligned}$$

as required.

Question 15 (c) (iii)

We proved that

$$\sum_{m=0}^{n}\frac{(-1)^m}{x+m}\binom{n}{m}=\frac{n!}{x(x+1)...(x+n)}$$

Let $x=1$.

$$\begin{aligned}\sum_{m=0}^{n}\frac{(-1)^m}{1+m}\binom{n}{m}&=\frac{n!}{1(2)...(1+n)}\\&=\frac{n!}{(n+1)!}\\&=\frac{1}{n+1}\end{aligned}$$

Hence, the limiting sum is 0 since $\frac{1}{n+1}\to 0$ as n approaches infinity.

Question 16 (a) (i)

We may suppose without any loss of generality that $\theta>\alpha$.

In order for $1+z+w=0$ to hold, the imaginary components of z and w are necessarily negatives of each other. Since they also have the same modulus, then z and w are complex conjugates.

$$\begin{aligned}1+z+\overline{z}&=0\\1+2\,\mathrm{Re}(z)&=0\\\mathrm{Re}(z)&=-\frac{1}{2}\end{aligned}$$

Hence $\cos\theta=-\frac{1}{2}$ and thus $\theta=\frac{2\pi}{3}$, since $\theta>\alpha$.

Therefore $z=\mathrm{cis}\left(\frac{2\pi}{3}\right)$ and $w=\mathrm{cis}\left(-\frac{2\pi}{3}\right)$, which forms an equilateral triangle with $z=1$.

Question 16 (a) (ii)

We are given that

$$2i + z_1 + z_2 = 0$$

Divide both sides by $2i$ so that we may acquire a result in the form of (i)

$$1 + \frac{z_1}{2i} + \frac{z_2}{2i} = 0$$

We are also given that $|2i| = |z_1| = |z_2|$, which is equivalent to

$$1 = \left|\frac{z_1}{2i}\right| = \left|\frac{z_2}{2i}\right|$$

This is exactly what we had in (i), but with $z = \frac{z_1}{2i}$ and $w = \frac{z_2}{2i}$. Hence, we may deduce that 1, $\frac{z_1}{2i}$ and $\frac{z_2}{2i}$ form an equilateral triangle in the Argand diagram.

But if we multiply all of the complex numbers by $2i$, then this has the effect of rotating them $\frac{\pi}{2}$ radians anti-clockwise about the origin, and scaling each side length of the triangle by a factor of 2. This does *not* change the fact that we still have an equilateral triangle, hence the result.

Question 16 (b) (i)

Since *u* and *v* form an equilateral triangle, they must be 60° rotations from each other about the origin, and hence we have $u = v \times \text{cis}\left(\pm\frac{\pi}{3}\right)$. The plus/minus covers both cases, depending on the orientation of the triangle.

Cube both sides.

$$\begin{aligned} u^3 &= v^3 \times \text{cis}(\pi) \\ &= -v^3 \\ u^3 + v^3 &= 0 \\ (u+v)(u^2 - uv + v^2) &= 0 \end{aligned}$$

It cannot be the case that $u = -v$, otherwise they would not form an equilateral triangle.

Hence, it must be the case that

$$\begin{aligned} u^2 - uv + v^2 &= 0 \\ u^2 + v^2 &= uv \end{aligned}$$

as required.

Question 16 (b) (ii)

Just choose simplest case when they are conjugates, $u = \text{cis}\left(\frac{\pi}{6}\right)$ and $v = \text{cis}\left(-\frac{\pi}{6}\right)$.

Question 16 (c) (i)

There are $(n-1)$ possible other people that Tom can have the hat of, and of course visa versa. To keep the sentences concise, let this other person be Jerry (note that Jerry is a 'representative' for any of the other $(n-1)$ people, and is not a unique person). This means that other than Tom and Jerry, there are $(n-2)$ other people, and hence there are $D(n-2)$ ways so that the remaining $(n-2)$ people all have the wrong hat. Hence, in total the number of ways that all *n* people have the wrong hat, where Tom and Jerry have each other's hat, is $(n-1)D(n-2)$.

Question 16 (c) (ii)

For any derangement, there most certainly exists a person who has somebody else's hat (by definition of the derangement). Let this person be Tom. Suppose Tom has somebody else's hat, in this case he has Jerry's hat.

Two situations can arrive from here.

Case #1: *Tom has Jerry's hat, and Jerry has Tom's hat.*

As found in (i), this can occur in $(n-1)D(n-2)$ ways.

Case #2: *Tom has Jerry's hat, but Jerry has somebody else's hat.*

There are $(n-1)$ possible "Jerrys". But since Jerry has somebody *else's* hat, that means that Jerry is now actually part of the $(n-1)$ remaining people who also have wrong hats. Hence, this scenario can occur in $(n-1)D(n-1)$ ways.

Adding the results, we have the required recurrence $D(n)=(n-1)\left[D(n-1)+D(n-2)\right]$.

Question 16 (c) (iii)

The required result can be acquired through algebraic manipulation of (ii),

$$\begin{aligned} D(n) &= (n-1)\left[D(n-1)+D(n-2)\right] \\ &= (n-1)D(n-1)+(n-1)D(n-2) \\ &= nD(n-1)-D(n-1)+(n-1)D(n-2) \end{aligned}$$

$$\begin{aligned} D(n)-nD(n-1) &= -D(n-1)+(n-1)D(n-2) \\ &= -\left[D(n-1)-(n-1)D(n-2)\right] \end{aligned}$$

Question 16 (c) (iv)

For simplicity, let $T_n = D(n) - nD(n-1)$.

This means that the result in (iii) is actually

$$\underbrace{D(n) - nD(n-1)}_{T_n} = -\underbrace{\left[D(n-1) - (n-1)D(n-2)\right]}_{T_{n-1}}$$

$$T_n = (-1)T_{n-1}$$

Substituting this in recursively, we eventually have

$$T_n = (-1)^{n-2} T_2$$

But

$$\begin{aligned} T_2 &= D(2) - 2D(1) \\ &= 1 - 2 \times 0 \\ &= 1 \end{aligned}$$

Hence, $T_n = (-1)^{n-2} = (-1)^n$, as required.

Question 16 (c) (v)

We will use the recurrence in (iv), which is a first-order recurrence. Hence, just the one base case is sufficient. Alternatively, one may use the second order recurrence in (ii), but then strong induction is required.

Base Case: $n=1$.

$$D(1)=1!\left(\frac{(-1)^0}{0!}+\frac{(-1)^1}{1!}\right)=0$$

And this agrees with the given initial value, so the base case is true.

Inductive Hypothesis: $n=k$

$$D(k)=k!\left(\frac{(-1)^0}{0!}+\frac{(-1)^1}{1!}+\ldots+\frac{(-1)^k}{k!}\right)$$

Inductive Step: $n=k \Rightarrow n=k+1$

Required to prove

$$D(k+1)=(k+1)!\left(\frac{(-1)^0}{0!}+\frac{(-1)^1}{1!}+\ldots+\frac{(-1)^{k+1}}{(k+1)!}\right)$$

We begin with the left hand side.

$$\begin{aligned}
D(k+1)&=(-1)^{k+1}+(k+1)D(k)\\
&=(-1)^{k+1}+(k+1)\times\underbrace{k!\left(\frac{(-1)^0}{0!}+\frac{(-1)^1}{1!}+\ldots+\frac{(-1)^k}{k!}\right)}_{D(k)\text{ from the inductive hypothesis}}\\
&=(-1)^{k+1}+(k+1)!\left(\frac{(-1)^0}{0!}+\frac{(-1)^1}{1!}+\ldots+\frac{(-1)^k}{k!}\right)\\
&=(k+1)!\left[\frac{(-1)^{k+1}}{(k+1)!}+\frac{(-1)^0}{0!}+\frac{(-1)^1}{1!}+\ldots+\frac{(-1)^k}{k!}\right]\\
&=(k+1)!\left(\frac{(-1)^0}{0!}+\frac{(-1)^1}{1!}+\ldots+\frac{(-1)^{k+1}}{(k+1)!}\right)
\end{aligned}$$

as required.