2010 NSW BOS Mathematics Extension 2 Solutions

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Q1a Let $u = 1+3x^2$, $\frac{1}{6}\frac{du}{dx} = x$

$$\int \frac{x}{\sqrt{1+3x^2}}dx = \int \frac{1}{6u^{\frac{1}{2}}}\frac{du}{dx}dx = \int \frac{1}{6u^{\frac{1}{2}}}du$$

$$= \frac{1}{3}u^{\frac{1}{2}} + c = \frac{1}{3}\sqrt{1+3x^2} + c$$

Q1b Let $u = \cos x$, $-\frac{du}{dx} = \sin x$

$$\int_0^{\frac{\pi}{4}} \tan x dx = \int_0^{\frac{\pi}{4}} \frac{\sin x}{\cos x}dx = \int_0^{\frac{\pi}{4}} -\frac{1}{u}\frac{du}{dx}dx = \int_1^{\frac{1}{\sqrt{2}}} -\frac{1}{u}du$$

$$\left[-\ln u\right]_1^{\frac{1}{\sqrt{2}}} = \ln\sqrt{2} = \frac{1}{2}\ln 2$$

Q1c $\int \frac{1}{x\left(x^2+1\right)}dx = \int\left(\frac{1}{x} - \frac{x}{x^2+1}\right)dx$ [Partial fractions]

$$= \ln|x| - \frac{1}{2}\ln\left(x^2+1\right) + c = \ln\left|\frac{x}{\sqrt{x^2+1}}\right| + c$$

Q1d $t = \tan\frac{x}{2}$, $\frac{dt}{dx} = \frac{1}{2}\sec^2\frac{x}{2} = \frac{1}{2}\left(1+\tan^2\frac{x}{2}\right)$, $2\frac{dt}{dx} = 1+t^2$

$\tan x = \frac{2\tan\frac{x}{2}}{1-\tan^2\frac{x}{2}} = \frac{2t}{1-t^2}$, $\sin x = \frac{2t}{1+t^2}$,

$$1+\sin x = 1+\frac{2t}{1+t^2} = \frac{(1+t)^2}{1+t^2}$$

$$\int_0^{\frac{\pi}{2}} \frac{dx}{1+\sin x} = \int_0^{\frac{\pi}{2}} \frac{\left(1+t^2\right)dx}{(1+t)^2} = \int_0^{\frac{\pi}{2}} \frac{2}{(1+t)^2}\frac{dt}{dx}dx$$

$$= \int_0^1 \frac{2dt}{(1+t)^2} = \left[-\frac{2}{1+t}\right]_0^1 = 1$$

Q1e Let $u = \sqrt{x}$, $u^2 = x$, $\frac{dx}{du} = 2u$, $2u\frac{du}{dx} = 1$

$$\int \frac{1}{1+\sqrt{x}}dx = \int \frac{1}{1+u}2u\frac{du}{dx}dx = \int \frac{1}{1+u}2udu = 2\int \frac{u}{1+u}du$$

$$= 2\int\left(1-\frac{1}{1+u}\right)du = 2u - 2\ln(1+u) + c = 2\sqrt{x} - 2\ln\left(1+\sqrt{x}\right) + c$$

Q2ai $z^2 = (5-i)^2 = 25-10i+i^2 = 24-10i$

Q2aii $z + 2\bar{z} = 5-i+2(5+i) = 15+i$

Q2aiii $\frac{i}{z} = \frac{i\bar{z}}{z\bar{z}} = \frac{i(5+i)}{26} = -\frac{1}{26}+\frac{5}{26}i$

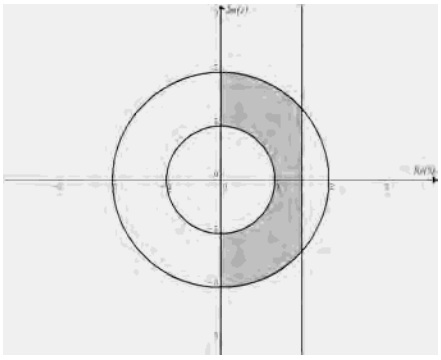
Q2bi $-\sqrt{3}-i$ is in the third quadrant.

$r = \sqrt{\left(-\sqrt{3}\right)^2+(-1)^2} = 2$, $\theta = \tan^{-1}\left(\frac{-1}{-\sqrt{3}}\right) = -\frac{5\pi}{6}$

$$-\sqrt{3}-i = 2cis\left(-\frac{5\pi}{6}\right)$$

Q2bii $\left(-\sqrt{3}-i\right)^6 = 2^6 cis\left(-\frac{5\pi}{6}\times 6\right) = 64cis\pi = -64 \in R$

Q2c $1 \le |z| \le 2$ is the region between two concentric circles of radii 1 and 2 and centred at O.
$0 \le z+\bar{z} \le 3$, $0 \le 2x \le 3$, $0 \le x \le 1.5$ is the region between vertical lines $x = 0$ and $x = 1.5$.



Q2di $\overline{OA} = |z| = |\cos\theta + i\sin\theta| = 1$

$\overline{OB} = \left|z^2\right| = |\cos 2\theta + i\sin 2\theta| = 1$

Since $OACB$ is a parallelogram, $\therefore \overline{BC} = \overline{OA} = 1$ and $\overline{AC} = \overline{OB} = 1$.
$\therefore$ $OACB$ is a rhombus.

Q2dii $\angle AOB = \arg z^2 - \arg z = 2\theta - \theta = \theta$

$OACB$ is a rhombus, $\therefore$ OC bisects $\angle AOB$, $\therefore$ $\angle COA = \frac{\theta}{2}$

$\therefore \arg\left(z+z^2\right) = \theta + \frac{\theta}{2} = \frac{3\theta}{2}$

Q2diii Let AB intersects OC at D. $OACB$ is a rhombus, $\therefore$ AD is a perpendicular bisector of OC.
$\therefore$ ΔOAD is a right-angled triangle.
$\therefore \overline{OD} = \cos\frac{\theta}{2}$, $\left|z+z^2\right| = \overline{OC} = 2\overline{OD} = 2\cos\frac{\theta}{2}$.

Q2div $\text{Re}\, z + \text{Re}\, z^2 = \cos\theta + \cos 2\theta$

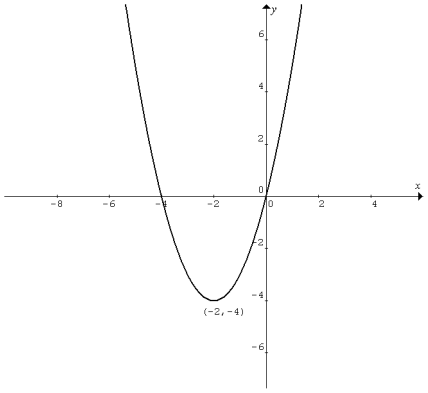
$$\text{Re}\left(z+z^2\right) = \left|z+z^2\right|\cos\frac{3\theta}{2} = 2\cos\frac{\theta}{2}\cos\frac{3\theta}{2}$$

Since $\text{Re}\, z + \text{Re}\, z^2 = \text{Re}\left(z+z^2\right)$,

$$\therefore \cos\theta + \cos 2\theta = 2\cos\frac{\theta}{2}\cos\frac{3\theta}{2}$$

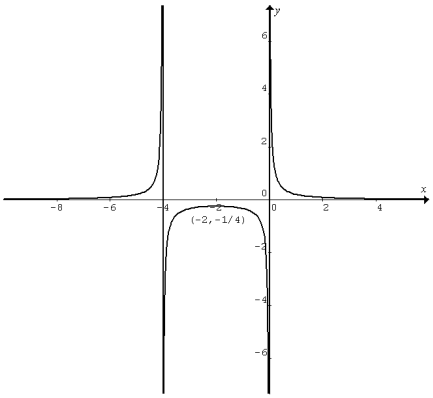
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Q3ai $y = x^2 + 4x = x(x+4)$

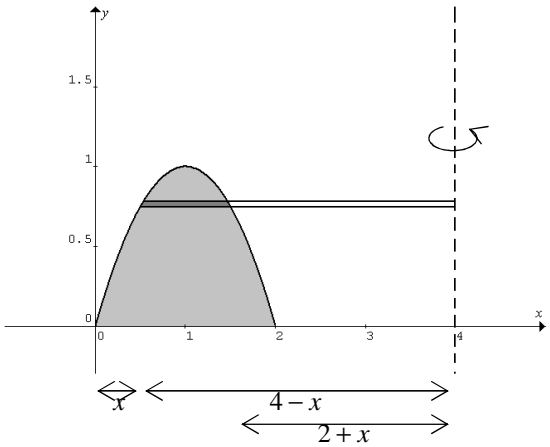


Q3aii $y = \dfrac{1}{x^2+4x} = \dfrac{1}{x(x+4)}$

Asymptotes: $x = 0$, $x = -4$, $y = 0$



Q3b



where $0 \le x \le 1$

$y = 2x - x^2$, $x = 1 - \sqrt{1-y}$

$$V = \int_0^1 \left(\pi(4-x)^2 - \pi(2+x)^2\right)dy = 12\pi\int_0^1 (1-x)dy$$

$$= 12\pi\int_0^1 \sqrt{1-y}\,dy = 12\pi\left[-\frac{2(1-y)^{\frac{3}{2}}}{3}\right]_0^1 = 8\pi$$

Q3c Let $\Pr(head) = p$ and $\Pr(tail) = 1-p$, given $p > 1-p$, i.e. $p > 0.5$

$\Pr(1head + 1tail) = \Pr(HT) + \Pr(TH) = 2p(1-p) = 0.48$

$\therefore\ p(1-p) = 0.24$, $\therefore\ p = 0.6$

$\Pr(2heads) = 0.6 \times 0.6 = 0.36$

Q3di Gradient $QA = \dfrac{c + \frac{c}{t}}{c + ct} = \dfrac{1}{t}$, .: gradient $\ell_1 = -t$

ℓ_1 is $y - \dfrac{c}{t} = -t(x - ct)$, i.e. $y = -tx + ct^2 + \dfrac{c}{t}$ ..... (1)

Q3dii Gradient $RA = \dfrac{c - \frac{c}{t}}{c - ct} = -\dfrac{1}{t}$, .: gradient $\ell_2 = t$

ℓ_2 is $y + \dfrac{c}{t} = t(x + ct)$, i.e. $y = tx + ct^2 - \dfrac{c}{t}$ ..... (2)

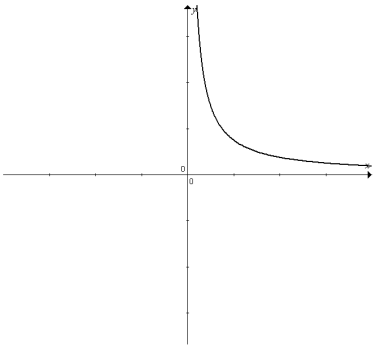
Q3diii Solve equations (1) and (2) simultaneously

$tx + ct^2 - \dfrac{c}{t} = -tx + ct^2 + \dfrac{c}{t}$, $2tx = 2\dfrac{c}{t}$, $x = \dfrac{c}{t^2}$, $y = ct^2$

Hence P is the point $\left(\dfrac{c}{t^2}, ct^2\right)$.

Q3div $x = \dfrac{c}{t^2}$, $y = ct^2$, $\therefore\ y = c\left(\dfrac{c}{x}\right)$, $\therefore\ xy = c^2$.

Since $x = \dfrac{c}{t^2} > 0$, the locus is $xy = c^2$ and $x \in R^+$, i.e. the half of the given rectangular hyperbola in the first quadrant.



Q4ai $\sqrt{x} + \sqrt{y} = 1$

Implicit differentiation: $\dfrac{1}{2\sqrt{x}} + \dfrac{1}{2\sqrt{y}}\dfrac{dy}{dx} = 0$, $\therefore\ \dfrac{dy}{dx} = -\sqrt{\dfrac{y}{x}}$

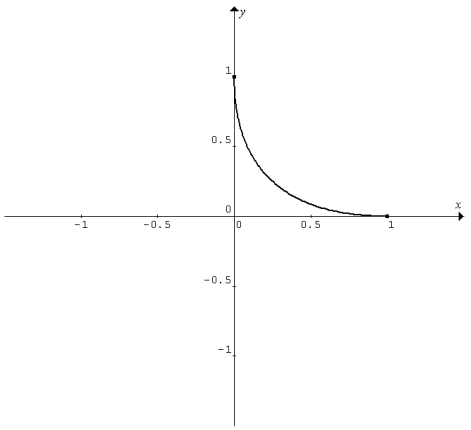
Q4aii $\sqrt{x} + \sqrt{y} = 1$ is defined for $x \in [0,1]$ and $y \in [0,1]$

At $x = 0$, $y = 1$, $\dfrac{dy}{dx} \to -\infty$

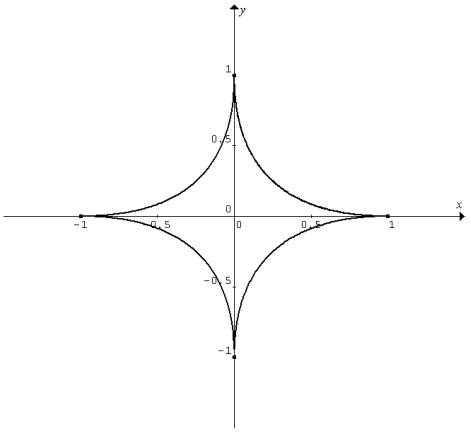
At $y = 0$, $x = 1$, $\dfrac{dy}{dx} = 0$

$\dfrac{dy}{dx}$ is negative for $x \in (0,1)$

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Q4aiii



Q4bi Resolving forces vertically:

$N\cos\alpha + F\sin\alpha - mg = 0$ ..... (1)

Resolving forces horizontally:

$N\sin\alpha - F\cos\alpha = \dfrac{mv^2}{r}$ ..... (2)

From (1), $N = \dfrac{mg - F\sin\alpha}{\cos\alpha}$ ..... (3)

Substitute (3) in (2), $\left(\dfrac{mg - F\sin\alpha}{\cos\alpha}\right)\sin\alpha - F\cos\alpha = \dfrac{mv^2}{r}$

Simplify to $F = mg\sin\alpha - \dfrac{mv^2}{r}\cos\alpha$

Q4bii $mg\sin\alpha - \dfrac{mv^2}{r}\cos\alpha = 0$, $v = \sqrt{gr\tan\alpha}$

Q4c Show $k \geq 4$ if $\dfrac{1}{a} + \dfrac{1}{b} = \dfrac{k}{a+b}$ for $a, b \in R^+$.

$\dfrac{1}{a} + \dfrac{1}{b} = \dfrac{k}{a+b}$, $\therefore (a+b)^2 = kab$ and $k > 0$

Let $a = nb$ where $n \in R^+$.

$\therefore (nb+b)^2 = knb^2$, $(n+1)^2 = kn$, $n^2 + (2-k)n + 1 = 0$

$n \in R^+$ exists when $\Delta = (2-k)^2 - 4 \geq 0$, $\therefore k \geq 4$ if $k > 0$.

Q4di Select 4 from 12: ${}^{12}C_4 = 495$

Q4dii There are ${}^{12}C_4$ ways to form the first group. For each of these there are ${}^{8}C_4$ ways to form the second group and one way to form the last group. The number of ways in this counting method is over counted by a factor of 3! because it takes into account the ordering of the groups.

Hence the number of ways $= \dfrac{{}^{12}C_4 \times {}^{8}C_4}{3!} = 5775$

Q5ai $B(b\cos\theta, b\sin\theta)$

Q5aii $P(a\cos\theta, b\sin\theta)$, $x = a\cos\theta$, $y = b\sin\theta$

$\therefore \dfrac{x}{a} = \cos\theta$ and $\dfrac{y}{b} = \sin\theta$

$\therefore \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = \cos^2\theta + \sin^2\theta = 1$

Q5aiii At $P(a\cos\theta, b\sin\theta)$, $\dfrac{dx}{d\theta} = -a\sin\theta$, $\dfrac{dy}{d\theta} = b\cos\theta$

Gradient of the tangent at $P = \dfrac{dy}{dx} = \dfrac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = -\dfrac{b\cos\theta}{a\sin\theta}$

Equation of the tangent at P:

$y - b\sin\theta = -\dfrac{b\cos\theta}{a\sin\theta}(x - a\cos\theta)$

$(a\sin\theta)y - ab\sin^2\theta = -(b\cos\theta)x + ab\cos^2\theta$

$(b\cos\theta)x + (a\sin\theta)y = ab$ ..... (1)

Q5aiv Change b to a in (1) to obtain the equation of the tangent to the circle at A:

$(a\cos\theta)x + (a\sin\theta)y = a^2$ ..... (2)

Eliminate x from (1) and (2): $a(1) - b(2)$

$(a^2\sin\theta)y - (ab\sin\theta)y = 0$, $\therefore (a^2\sin\theta - ab\sin\theta)y = 0$

Since $a > b$, $\therefore a^2\sin\theta - ab\sin\theta \neq 0$ and $y = 0$

$\therefore$ the two tangents intersect at a point on the x-axis.

Q5b $\displaystyle\int \frac{dy}{y(1-y)} = \int\left(\frac{1}{y} + \frac{1}{1-y}\right)dy = \ln y - \ln(1-y) + c$

$= \ln\left(\dfrac{y}{1-y}\right) + c$ where $0 < y < 1$

Q5ci $\dfrac{dy}{dx} = ay(1-y)$ is an inverted parabola and its horizontal axis intercepts are $(0,0)$ and $(1,0)$. Its axis of symmetry is $y = \dfrac{0+1}{2} = \dfrac{1}{2}$. $\therefore \dfrac{dy}{dx}$ has its maximum value when $y = \dfrac{1}{2}$.

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Q5cii $\dfrac{dy}{dx}=ay(1-y)$, $\dfrac{dx}{dy}=\dfrac{1}{ay(1-y)}$, $x=\dfrac{1}{a}\int\dfrac{dy}{y(1-y)}$

$\therefore\ ax=\ln\left(\dfrac{y}{1-y}\right)+c$, $\dfrac{y}{1-y}=e^{ax-c}$, $y=(1-y)e^{ax-c}$,

$\left(1+e^{ax-c}\right)y=e^{ax-c}$, $\therefore\ y=\dfrac{e^{ax-c}}{1+e^{ax-c}}=\dfrac{1}{e^{-ax+c}+1}$

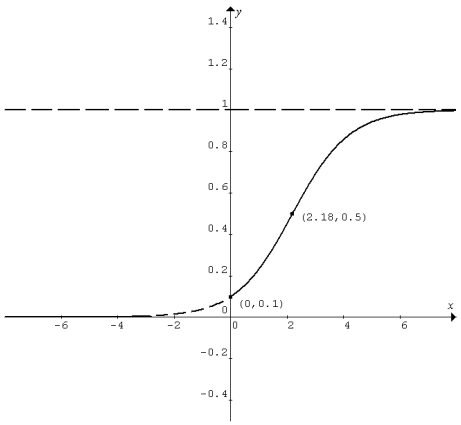
$\therefore\ y=\dfrac{1}{ke^{-ax}+1}$ where $k=e^c$

Q5ciii Given $y(0)=0.1$, $\therefore\ y=\dfrac{1}{k+1}=0.1$, $\therefore\ k=9$

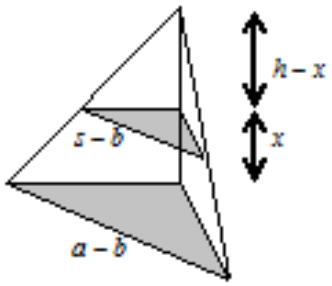
Q5civ As x increases from 0, the gradient of $y=\dfrac{1}{ke^{-ax}+1}$ increases to a maximum value at $y=\dfrac{1}{2}$ and then decreases,

$\therefore$ there is a point of inflection at $y=\dfrac{1}{2}$.

Q5cv $y=\dfrac{1}{9e^{-ax}+1}$, $x\ge 0$



Q6ai Place two sloping edges together:



Two similar triangular solids: $\dfrac{s-b}{a-b}=\dfrac{h-x}{h}=1-\dfrac{x}{h}$,

$\therefore\ s-b=a-b-\dfrac{(a-b)}{h}x$, $s=a-\dfrac{(a-b)}{h}x$

Q6aii At height x m, area of horizontal layer

$=s^2=\left(a-\dfrac{(a-b)}{h}x\right)^2$ m^2

Volume of frustum $=\displaystyle\int_0^h\left(a-\frac{(a-b)}{h}x\right)^2dx$

$=\left[-\dfrac{h}{3(a-b)}\left(a-\dfrac{(a-b)}{h}x\right)^3\right]_0^h$

$=-\dfrac{h}{3(a-b)}\left(b^3-a^3\right)=\dfrac{h}{3(a-b)}\left(a^3-b^3\right)$

$=\dfrac{h}{3}\left(a^2+ab+b^2\right)$ m^3

Q6b $a_n=2a_{n-1}+a_{n-2}$, $n\ge 2$, $a_0=a_1=2$

$a_n=\left(1+\sqrt{2}\right)^n+\left(1-\sqrt{2}\right)^n$ is true when $n=0$ because

$a_0=\left(1+\sqrt{2}\right)^0+\left(1-\sqrt{2}\right)^0=2$

It is also true when $n=1$ because $a_1=\left(1+\sqrt{2}\right)^1+\left(1-\sqrt{2}\right)^1=2$

Assuming that it is true when $n=k-1$ and $n=k-2$, where $k\ge 2$, then $a_k=2a_{k-1}+a_{k-2}$

$=2\left(1+\sqrt{2}\right)^{k-1}+2\left(1-\sqrt{2}\right)^{k-1}+\left(1+\sqrt{2}\right)^{k-2}+\left(1-\sqrt{2}\right)^{k-2}$

$=2\left(1+\sqrt{2}\right)^{k-1}+\left(1+\sqrt{2}\right)^{k-2}+2\left(1-\sqrt{2}\right)^{k-1}+\left(1-\sqrt{2}\right)^{k-2}$

$=\left(1+\sqrt{2}\right)^{k-2}\left(2\left(1+\sqrt{2}\right)+1\right)+\left(1-\sqrt{2}\right)^{k-2}\left(2\left(1-\sqrt{2}\right)+1\right)$

$=\dfrac{\left(1+\sqrt{2}\right)^k}{3+2\sqrt{2}}\left(3+2\sqrt{2}\right)+\dfrac{\left(1-\sqrt{2}\right)^k}{3-2\sqrt{2}}\left(3-2\sqrt{2}\right)$

$=\left(1+\sqrt{2}\right)^k+\left(1-\sqrt{2}\right)^k$

$\therefore$ it is also true for $n=k$

$\therefore$ it is true for all $n\ge 0$

Q6ci

$(\cos\theta+i\sin\theta)^5=\cos^5\theta+5\cos^4\theta(i\sin\theta)+10\cos^3\theta(i\sin\theta)^2$

$+10\cos^2\theta(i\sin\theta)^3+5\cos\theta(i\sin\theta)^4+(i\sin\theta)^5$

$=\cos^5\theta+i5\cos^4\theta\sin\theta-10\cos^3\theta\sin^2\theta$

$-i10\cos^2\theta\sin^3\theta+5\cos\theta\sin^4\theta+i\sin^5\theta$

$=\left(\cos^5\theta-10\cos^3\theta\sin^2\theta+5\cos\theta\sin^4\theta\right)$

$+i\left(5\cos^4\theta\sin\theta-10\cos^2\theta\sin^3\theta+\sin^5\theta\right)$

Q6cii $(\cos\theta+i\sin\theta)^5=\cos 5\theta+i\sin 5\theta$

$\therefore\ \sin 5\theta=5\cos^4\theta\sin\theta-10\cos^2\theta\sin^3\theta+\sin^5\theta$

$=5\left(1-\sin^2\theta\right)^2\sin\theta-10\left(1-\sin^2\theta\right)\sin^3\theta+\sin^5\theta$

$=5\left(1-2\sin^2\theta+\sin^4\theta\right)\sin\theta-10\left(1-\sin^2\theta\right)\sin^3\theta+\sin^5\theta$

$=16\sin^5\theta-20\sin^3\theta+5\sin\theta$

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Q6ciii $16\sin^5\theta - 20\sin^3\theta + 5\sin\theta = \sin 5\theta$ Let $\theta = \frac{\pi}{10}$,

$$16\left(\sin\left(\frac{\pi}{10}\right)\right)^5 - 20\left(\sin\left(\frac{\pi}{10}\right)\right)^3 + 5\sin\left(\frac{\pi}{10}\right) = \sin\left(\frac{\pi}{2}\right)$$

$$\therefore 16\left(\sin\left(\frac{\pi}{10}\right)\right)^5 - 20\left(\sin\left(\frac{\pi}{10}\right)\right)^3 + 5\sin\left(\frac{\pi}{10}\right) - 1 = 0$$

$\therefore x = \sin\left(\frac{\pi}{10}\right)$ is one of the solutions to $16x^5 - 20x^3 + 5x - 1 = 0$

Q6civ $16x^5 - 20x^3 + 5x - 1 = (x-1)(16x^4 + bx^3 + cx^2 + dx + 1)$

$\therefore -b + c = -20$, $-c + d = 0$ and $-d + 1 = 5$

$\therefore d = -4$, $c = -4$ and $b = 16$

$\therefore p(x) = 16x^4 + 16x^3 - 4x^2 - 4x + 1$

Q6cv $p(x) = 16x^4 + 16x^3 - 4x^2 - 4x + 1 = (4x^2 + ax - 1)^2$

Compare the coefficients of x: $-2a = -4$, $a = 2$

Q6cvi $16x^5 - 20x^3 + 5x - 1 = (x-1)(4x^2 + 2x - 1)^2 = 0$

Since $x - 1 = \sin\left(\frac{\pi}{10}\right) - 1 \neq 0$, $\therefore 4x^2 + 2x - 1 = 0$

$$\therefore x = \sin\left(\frac{\pi}{10}\right) = \frac{-2 + \sqrt{4+16}}{8} = \frac{-1+\sqrt{5}}{4} \text{ (first quadrant)}$$

Q7ai $\angle ABD = \angle ACD$ (angles subtended on the circumference by the same arc are equal)

$\angle ADK = \angle CDB$ (given)

$\therefore \angle ADK + \angle KDB = \angle KDB + \angle CDB$

$\therefore \angle ADB = \angle KDC$

$\therefore \Delta ADB, \Delta KDC$ are similar

Q7aii $\Delta ADB, \Delta KDC$ are similar

$\therefore \frac{DC}{BD} = \frac{KC}{AB}$, $\therefore BD \times KC = AB \times DC$ ..... (1)

$\Delta ADK, \Delta BDC$ are similar

$\therefore \frac{AD}{BD} = \frac{AK}{BC}$, $\therefore BD \times AK = AD \times BC$ ..... (2)

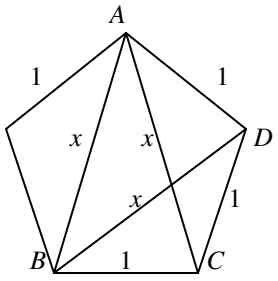
Add (1) and (2):

$BD \times AK + BD \times KC = AD \times BC + AB \times DC$

$BD \times (AK + KC) = AD \times BC + AB \times DC$

$\therefore BD \times AC = AD \times BC + AB \times DC$

Q7aiii

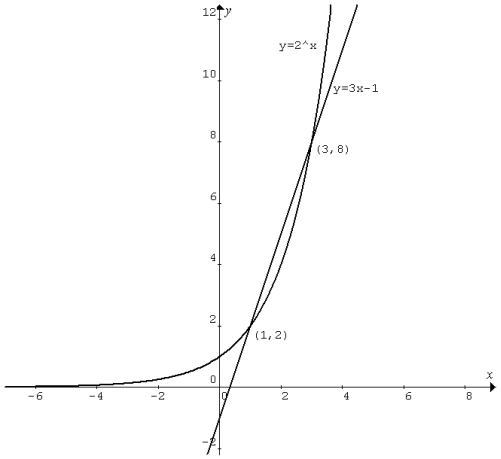


Since $BD \times AC = AD \times BC + AB \times DC$

$\therefore x^2 = 1 + x$ where $x > 0$, $x^2 - x - 1 = 0$

$$x = \frac{1+\sqrt{1+4}}{2} = \frac{1+\sqrt{5}}{2}$$

Q7b $2^x \geq 3x - 1$ for $x \geq 3$



Q7ci $P(x) = (n-1)x^n - nx^{n-1} + 1$, $n = 3,5,7,.....$

Let $P'(x) = n(n-1)x^{n-1} - n(n-1)x^{n-2} = 0$

$\therefore P'(x) = n(n-1)x^{n-2}(x-1) = 0$

$\therefore x = 0$ or $x = 1$

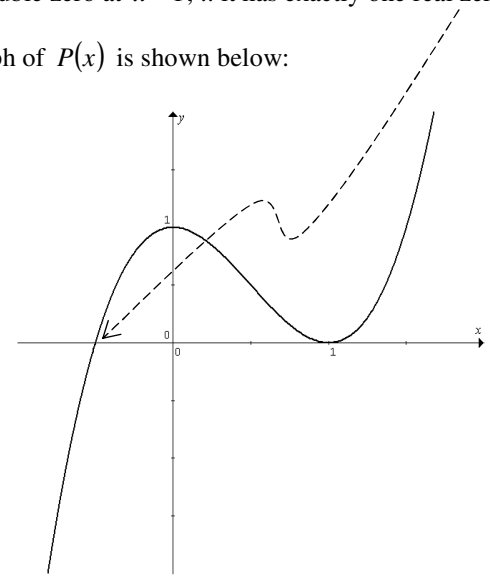
$\therefore P(x)$ has exactly two stationary points.

Q7cii $P(1) = n - 1 - n + 1 = 0$ and $(1,0)$ is a stationary point,

$\therefore P(x)$ has a double zero at $x = 1$

Q7ciii $P(x)$ is an odd-degree polynomial with 2 turning points and a double zero at $x = 1$, $\therefore$ it has exactly one real zero other than 1.

The graph of $P(x)$ is shown below:



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Q7civ $P(x)=(n-1)x^n-nx^{n-1}+1$, $n=3,5,7,.....$

For $n=3$:

$P(x)=2x^3-3x^2+1=(x-1)^2(2x+1)$, $\therefore\ \alpha=-\frac{1}{2}$

For $n>3$:

$P(-1)=(n-1)(-1)^n-n(-1)^{n-1}+1=-n+1-n+1=2-2n<0$

$P\left(-\frac{1}{2}\right)=(n-1)\left(-\frac{1}{2}\right)^n-n\left(-\frac{1}{2}\right)^{n-1}+1=-\frac{n-1}{2^n}-\frac{n}{2^{n-1}}+1$

$=\frac{-n+1-2n}{2^n}+1=-\frac{3n-1}{2^n}+1$

From part b, for $n>3$, $2^n>3n-1$, $\therefore 1>\frac{3n-1}{2^n}$, $\therefore P\left(-\frac{1}{2}\right)>0$

$\therefore\ P(x)$ has an x-intercept in $\left(-1,-\frac{1}{2}\right]$.

Hence $-1<\alpha\le-\frac{1}{2}$.

Q7cv $4x^5-5x^4+1$ is in the form $P(x)=(n-1)x^n-nx^{n-1}+1$.

It has a real zero α between -1 and $-\frac{1}{2}$, and a double zero 1.

Each of these zeros has modulus ≤ 1. The polynomial is of degree 5 with real coefficients, $\therefore$ it also has a pair of complex zeros which are conjugates, they are $a+ib$ and $a-ib$.

Product of the zeros: $\alpha\times1^2\times(a+ib)(a-ib)=-\frac{1}{4}$

$\therefore\ a^2+b^2=-\frac{1}{4\alpha}$

Since $-1<\alpha\le-\frac{1}{2}$, $-4<4\alpha\le-2$, $2\le-4\alpha<4$,

$\frac{1}{4}<-\frac{1}{4\alpha}\le\frac{1}{2}$, $\therefore\ \frac{1}{4}<a^2+b^2\le\frac{1}{2}$, $\frac{1}{2}<\sqrt{a^2+b^2}\le\frac{1}{\sqrt{2}}$,

$\therefore\ \frac{1}{2}<|a\pm ib|\le\frac{1}{\sqrt{2}}$

Hence each of the zeros of $4x^5-5x^4+1$ has modulus ≤ 1.

Q8a Apply the reduction formula

$\int\cos^n x\,dx=\frac{1}{n}\cos^{n-1}x\sin x+\frac{n-1}{n}\int\cos^{n-2}x\,dx$ to A_n

$A_n=\int_0^{\frac{\pi}{2}}\cos^{2n}x\,dx=\left[\frac{1}{2n}\cos^{2n-1}x\sin x\right]_0^{\frac{\pi}{2}}+\frac{2n-1}{2n}\int_0^{\frac{\pi}{2}}\cos^{2n-2}x\,dx$

$=\frac{2n-1}{2n}A_{n-1}$

$\therefore\ nA_n=\frac{2n-1}{2}A_{n-1}$ for $n\ge1$

OR integration by parts on A_n: $\left(\int u\frac{dv}{dx}dx=uv-\int v\frac{du}{dx}dx\right)$

Let $u=\cos^{2n-1}x$, $\frac{du}{dx}=-(2n-1)\sin x\cos^{2n-2}x$

Let $\frac{dv}{dx}=\cos x$, $v=\sin x$

$A_n=\int_0^{\frac{\pi}{2}}\cos^{2n}x\,dx$

$=\left[\sin x\cos^{2n-1}x\right]_0^{\frac{\pi}{2}}-\int_0^{\frac{\pi}{2}}\sin x\left(-(2n-1)\sin x\cos^{2n-2}x\right)dx$

$=(2n-1)\int_0^{\frac{\pi}{2}}(1-\cos^2x)\cos^{2n-2}x\,dx$

$=(2n-1)\int_0^{\frac{\pi}{2}}\cos^{2n-2}x\,dx-(2n-1)\int_0^{\frac{\pi}{2}}\cos^{2n}x\,dx$

$\therefore\ A_n=(2n-1)A_{n-1}-(2n-1)A_n$, $\therefore\ nA_n=\frac{2n-1}{2}A_{n-1}$ for $n\ge1$.

Q8b Integration by parts on $A_n=\int_0^{\frac{\pi}{2}}\cos^{2n}x\,dx$:

Let $u=\cos^{2n}x$, $\frac{du}{dx}=-2n\sin x\cos^{2n-1}x$

Let $\frac{dv}{dx}=1$, $v=x$

$A_n=\int_0^{\frac{\pi}{2}}\cos^{2n}x\,dx=\left[x\cos^{2n}x\right]_0^{\frac{\pi}{2}}-\int_0^{\frac{\pi}{2}}x\left(-2n\sin x\cos^{2n-1}x\right)dx$

$=2n\int_0^{\frac{\pi}{2}}x\sin x\cos^{2n-1}x\,dx$

Q8c Let $\frac{dv}{dx}=x$, $v=\frac{x^2}{2}$

Let $u=\sin x\cos^{2n-1}x$,

$\frac{du}{dx}=\cos x\cos^{2n-1}x+\sin x(2n-1)\cos^{2n-2}x(-\sin x)$

$=\cos^{2n}x-(2n-1)(1-\cos^2x)\cos^{2n-2}x$

$=\cos^{2n}x-(2n-1)\cos^{2n-2}x+(2n-1)\cos^{2n}x$

$=2n\cos^{2n}x-(2n-1)\cos^{2n-2}x$

$\therefore\ A_n=2n\int_0^{\frac{\pi}{2}}x\sin x\cos^{2n-1}x\,dx$

$=2n\left[\frac{x^2}{2}\sin x\cos^{2n-1}x\right]_0^{\frac{\pi}{2}}-2n\int_0^{\frac{\pi}{2}}\frac{x^2}{2}\left(2n\cos^{2n}x-(2n-1)\cos^{2n-2}x\right)dx$

$=n(2n-1)\int_0^{\frac{\pi}{2}}x^2\cos^{2n-2}x\,dx-2n^2\int_0^{\frac{\pi}{2}}x^2\cos^{2n}x\,dx$

$\therefore\ A_n=n(2n-1)B_{n-1}-2n^2B_n$ $\quad\therefore\ \frac{A_n}{n^2}=\frac{(2n-1)}{n}B_{n-1}-2B_n$

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Q8d From part c:

$\frac{A_n}{n^2}=\frac{(2n-1)}{n}B_{n-1}-2B_n$, $\therefore\ \frac{1}{n^2}=\frac{(2n-1)}{n}\frac{B_{n-1}}{A_n}-2\frac{B_n}{A_n}$ ..... (1)

From part a: $nA_n=\frac{(2n-1)}{2}A_{n-1}$, $\therefore\ \frac{2}{A_{n-1}}=\frac{(2n-1)}{nA_n}$ ..... (2)

Substitute (2) in (1): $\frac{1}{n^2}=2\left(\frac{B_{n-1}}{A_{n-1}}-\frac{B_n}{A_n}\right)$ for $n\geq 1$

Q8e

$$\sum_{k=1}^{n}\frac{1}{k^2}=2\left(\frac{B_0}{A_0}-\frac{B_1}{A_1}\right)+2\left(\frac{B_1}{A_1}-\frac{B_2}{A_2}\right)+2\left(\frac{B_2}{A_2}-\frac{B_3}{A_3}\right)+\ldots+2\left(\frac{B_{n-1}}{A_{n-1}}-\frac{B_n}{A_n}\right)$$

$$=2\left(\frac{B_0}{A_0}-\frac{B_n}{A_n}\right)$$

where $A_0=\int_0^{\frac{\pi}{2}}dx=\left[x\right]_0^{\frac{\pi}{2}}=\frac{\pi}{2}$ and $B_0=\int_0^{\frac{\pi}{2}}x^2dx=\left[\frac{x^3}{3}\right]_0^{\frac{\pi}{2}}=\frac{\pi^3}{24}$

$\therefore\ \sum_{k=1}^{n}\frac{1}{k^2}=2\left(\frac{B_0}{A_0}-\frac{B_n}{A_n}\right)=2\frac{B_0}{A_0}-2\frac{B_n}{A_n}=\frac{\pi^2}{6}-2\frac{B_n}{A_n}$

Q8f $\sin x\geq\frac{2x}{\pi}$, $\sin^2 x=1-\cos^2 x\geq\frac{4x^2}{\pi^2}$

$\therefore\ \cos^2 x\leq 1-\frac{4x^2}{\pi^2}$, $\therefore\ \cos^{2n}x\leq\left(1-\frac{4x^2}{\pi^2}\right)^n$ for $0\leq x\leq\frac{\pi}{2}$

Hence $B_n=\int_0^{\frac{\pi}{2}}x^2\cos^{2n}x\,dx\leq\int_0^{\frac{\pi}{2}}x^2\left(1-\frac{4x^2}{\pi^2}\right)^n dx$

Q8g Integration by parts:

Let $u=x$ and $\frac{dv}{dx}=x\left(1-\frac{4x^2}{\pi^2}\right)^n$

$\therefore\ \frac{du}{dx}=1$ and

$$v=\int x\left(1-\frac{4x^2}{\pi^2}\right)^n dx=\int\frac{-\pi^2p^n}{8}dp=-\frac{\pi^2}{8(n+1)}\left(1-\frac{4x^2}{\pi^2}\right)^{n+1}$$

where $p=1-\frac{4x^2}{\pi^2}$.

$$\int_0^{\frac{\pi}{2}}x^2\left(1-\frac{4x^2}{\pi^2}\right)^n dx$$

$$=\left[-\frac{\pi^2}{8(n+1)}x\left(1-\frac{4x^2}{\pi^2}\right)^{n+1}\right]_0^{\frac{\pi}{2}}-\int_0^{\frac{\pi}{2}}-\frac{\pi^2}{8(n+1)}\left(1-\frac{4x^2}{\pi^2}\right)^{n+1}dx$$

$$=\frac{\pi^2}{8(n+1)}\int_0^{\frac{\pi}{2}}\left(1-\frac{4x^2}{\pi^2}\right)^{n+1}dx$$

Q8h Let $x=\frac{\pi}{2}\sin t$.

$$\frac{\pi^2}{8(n+1)}\int_0^{\frac{\pi}{2}}\left(1-\frac{4x^2}{\pi^2}\right)^{n+1}dx=\frac{\pi^2}{8(n+1)}\int_0^{\frac{\pi}{2}}\left(1-\sin^2 t\right)^{n+1}\frac{\pi}{2}\cos t\,dt$$

$$=\frac{\pi^3}{16(n+1)}\int_0^{\frac{\pi}{2}}\left(\cos^2 t\right)^{n+1}\cos t\,dt=\frac{\pi^3}{16(n+1)}\int_0^{\frac{\pi}{2}}\cos^{2n+3}t\,dt$$

$\therefore\ B_n\leq\frac{\pi^3}{16(n+1)}\int_0^{\frac{\pi}{2}}\cos^{2n+3}t\,dt$ from parts (f) and (g).

For $0\leq t\leq\frac{\pi}{2}$, $0\leq\cos^3 t\leq 1$

$\therefore\ B_n\leq\frac{\pi^3}{16(n+1)}\int_0^{\frac{\pi}{2}}\cos^{2n+3}t\,dt\leq\frac{\pi^3}{16(n+1)}\int_0^{\frac{\pi}{2}}\cos^{2n}t\,dt$

$\therefore\ B_n\leq\frac{\pi^3}{16(n+1)}\int_0^{\frac{\pi}{2}}\cos^{2n+3}t\,dt\leq\frac{\pi^3}{16(n+1)}A_n$

Q8i Note that $A_n>0$ and $B_n>0$

From part (h), $2\frac{B_n}{A_n}\leq\frac{\pi^3}{8(n+1)}$

From part (e), $\frac{\pi^2}{6}-2\frac{B_n}{A_n}=\sum_{k=1}^{n}\frac{1}{k^2}$

$\therefore\ \frac{\pi^2}{6}-\frac{\pi^3}{8(n+1)}\leq\sum_{k=1}^{n}\frac{1}{k^2}<\frac{\pi^2}{6}$

Q8j The value of $\sum_{k=1}^{n}\frac{1}{k^2}$ lies between $\frac{\pi^2}{6}-\frac{\pi^3}{8(n+1)}$ and $\frac{\pi^2}{6}$.

As $n\to\infty$, $\frac{\pi^3}{8(n+1)}\to 0$, $\sum_{k=1}^{n}\frac{1}{k^2}\to\frac{\pi^2}{6}$

$\therefore\ \lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{k^2}=\frac{\pi^2}{6}$

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