4 Unit HSC (2004)

Question 1.

a) $\int xe^{3x}\,dx$

$= \int x \cdot \frac{d}{dx}\left(\frac{1}{3}e^{3x}\right) dx$

$= \left[x \cdot \frac{1}{3}e^{3x}\right] - \int \frac{1}{3}e^{3x} \cdot 1$

$= \frac{1}{3}xe^{3x} - \frac{1}{9}e^{3x} + C$ ✱

b) Let $u = \cos x$. $u' = -\sin x$

When $x = 0$, $u = 1$

$x = \frac{\pi}{4}$, $u = \frac{1}{\sqrt{2}}$

$\therefore \int_0^{\frac{\pi}{4}} \frac{\sin x}{\cos^3 x}\,dx$

$= \int_1^{\frac{1}{\sqrt{2}}} -u^{-3}\,du$

$= \left[\frac{1}{2}u^{-2}\right]_1^{1/\sqrt{2}}$

$= \frac{1}{2}\left[2 - 1\right]$

$= \frac{1}{2}$ ✱

c) $\int \frac{dx}{\sqrt{5 + 4x - x^2}}$

$= \int \frac{dx}{\sqrt{9 - (2-x)^2}}$

$= -\sin^{-1}\left(\frac{2-x}{3}\right) + C$ ✱

d) i) $\frac{x^2 - 7x + 4}{(x+1)(x-1)^2}$

$\equiv \frac{a(x^2 - 2x + 1) + b(x^2 - 1) - (x+1)}{(x+1)(x-1)^2}$

$a + b = 1$ ——(1)

$-2a - 1 = -7$ ——(2)

$\therefore -2a = -6$

$a = 3$

$\therefore b = -2$

ii) $\int \frac{x^2 - 7x + 4}{(x+1)(x-1)^2}$

$= \int \frac{3\,dx}{x+1} + \int \frac{-2}{(x-1)}\,dx - \int \frac{1}{(x-1)^2}\,dx$

$= 3\log(x+1) - 2\log(x-1) + \frac{1}{(x-1)}$ ✱

e) $\int_0^1 \frac{x^2}{\sqrt{4 - x^2}}\,dx$

Let $x = 2\sin\theta$

$\frac{dx}{d\theta} = 2\cos\theta$

$dx = 2\cos\theta\,d\theta$

When $x = 0$, $\theta = 0$

$x = 1$, $\theta = \frac{\pi}{6}$

$\therefore \int_0^1 \frac{x^2}{\sqrt{4 - x^2}}\,dx$

$= \int_0^{\frac{\pi}{6}} \frac{4\sin^2\theta}{\sqrt{4 - 4\sin^2\theta}} 2\cos\theta\,d\theta$

$= \int_0^{\frac{\pi}{6}} 4\sin^2\theta\,d\theta$

$= 4 \times \left[\frac{1}{2}\left(x - \frac{1}{2}\sin 2x\right)\right]_0^{\frac{\pi}{6}}$

$= 2\left[\left(\frac{\pi}{6} - \frac{1}{2}\sin\frac{\pi}{3}\right) - 0\right]$

$= \left[\frac{\pi}{3} - \sin\frac{\pi}{3}\right]$

$= \frac{\pi}{3} - \frac{\sqrt{3}}{2}$ ✱

Question 2

(i) (i) $zw = (1+2i)(3-i)$

$= 3 + 6i - i + 2 = 5 + 5i = 5(1+i)$

(ii) $\frac{10}{z} = \frac{10}{1+2i} \times \frac{1-2i}{1-2i}$

$= \frac{10-20i}{1+4} = 2 - 4i$

$\overline{\left(\frac{10}{z}\right)} = 2 + 4i$

(ii) (i) $\frac{\alpha}{\beta} = \frac{1+i\sqrt{3}}{1+i} \times \frac{1-i}{1-i}$

$= \frac{1+i\sqrt{3}-i+\sqrt{3}}{2} = \frac{1+\sqrt{3}}{2} + i\frac{(\sqrt{3}-1)}{2}$

(ii) $r = \sqrt{1^2 + (\sqrt{3})^2} = 2$

$\theta = \tan^{-1}(\sqrt{3}) = \frac{\pi}{3}$

$\alpha = 2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$

(iii) $\left|\frac{\alpha}{\beta}\right| = \frac{|\alpha|}{|\beta|} = \frac{2}{\sqrt{2}} = \sqrt{2}$

$\arg\left(\frac{\alpha}{\beta}\right) = \arg\alpha - \arg\beta = \frac{\pi}{3} - \frac{\pi}{4} = \frac{\pi}{12}$

$\frac{\alpha}{\beta} = \sqrt{2}\left(\cos\frac{\pi}{12} + i\sin\frac{\pi}{12}\right)$

(iv) Equating expressions from (i) and (iii) for $\frac{\alpha}{\beta}$ gives:

$\sqrt{2}\sin\frac{\pi}{12} = \frac{\sqrt{3}-1}{2}$

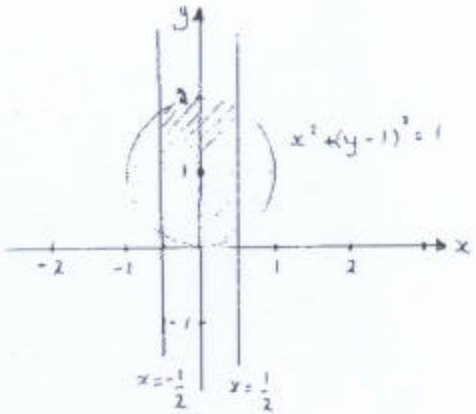
i.e. $\sin\left(\frac{\pi}{12}\right) = \frac{\sqrt{3}-1}{2\sqrt{2}}$

(i) $|z+\bar{z}| \le 1$ $\quad$ $|z-i| \le 1$

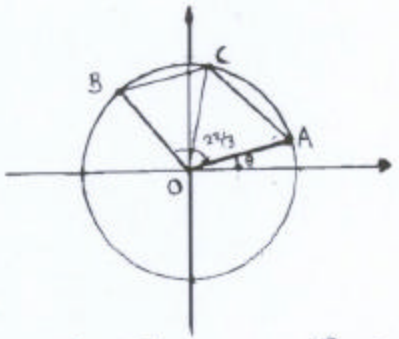
$|2x| \le 1$ $\quad$ $|x+iy-i| \le 1$

$-\frac{1}{2} \le x \le \frac{1}{2}$ $\quad$ $|x+i(y-1)| \le 1$

$x^2 + (y-1)^2 \le 1$



(d)



(i) $\vec{OA} + \vec{AC} = \vec{OC}$ i.e. $z + \vec{AC} = zw$ i.e. $\vec{AC} = w = \vec{OB}$

Also $OA = OB = OC$ (equal radii)

Hence ΔOAC is equilateral i.e. $\angle COA = \pi/3$

Similarly ΔOBC is equilateral i.e. $\angle BOC = \frac{\pi}{3}$

$\therefore \angle AOB = \frac{\pi}{3} + \frac{\pi}{3} = \frac{2\pi}{3}$

(ii) $z = r(\cos\theta + i\sin\theta)$, $z^3 = r^3(\cos 3\theta + i\sin 3\theta)$

$w = r\left[\cos\left(\theta + \frac{2\pi}{3}\right) + i\sin\left(\theta + \frac{2\pi}{3}\right)\right]$

$w^3 = r^3\left[\cos\left(3\theta + 2\pi\right) + i\sin\left(3\theta + 2\pi\right)\right]$

$= r^3(\cos 3\theta + i\sin 3\theta)$

$= z^3$

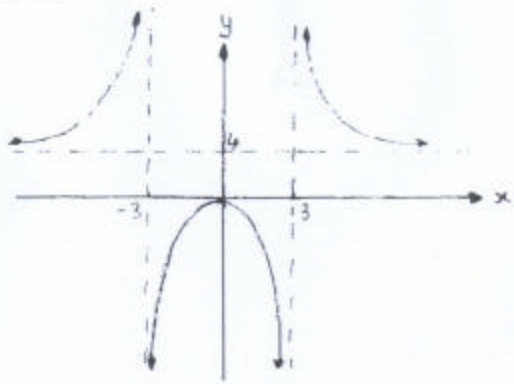
(iii) $z^3 - w^3 = 0$

$(z-w)(z^2 + zw + w^2) = 0$

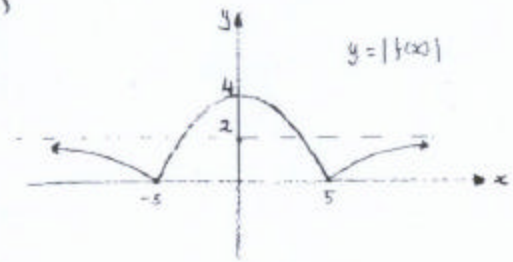
i.e. $z^2 + w^2 + zw = 0$

Question 3

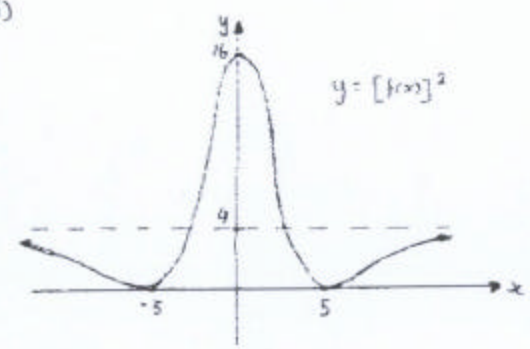
(ii)



(b) (i)



(ii)



(iii)



) $x^2 - xy + y^3 = 5$

Using implicit differentiation gives:

$2x - (x\cdot y' + y\cdot 1) + 3y^2 y' = 0$

$2x - y = xy' - 3y^2 y'$

$y' = \frac{2x - y}{x - 3y^2}$

At $(2, -1)$, $y' = \frac{4+1}{2-3} = -5$

Equation of tangent is given by:

$(y+1) = -5(x-2)$

$y + 1 = -5x + 10$

$y = -5x + 9$

(xi) At $x = h$, the length of each side of the triangle is $2h^4$.

Area of equilateral $\Delta = \frac{1}{2}ab\sin C$

$= \frac{1}{2} \times 2h^4 \times 2h^4 \times \sin 60^\circ$

$= 2h^8 \times \frac{\sqrt{3}}{2}$

$= \sqrt{3}h^8$

(ii) $V = \int_0^2 \sqrt{3}h^8 \, dh$

$= \sqrt{3}\left[\frac{h^9}{9}\right]_0^2 = \frac{512\sqrt{3}}{9}$ units^3

Question 4

) (i) $\alpha^2\beta\gamma + \alpha\beta^2\gamma + \alpha\beta\gamma^2 = \alpha\beta\gamma(\alpha + \beta + \gamma)$

$= -\frac{51}{3}\left(-\frac{7}{3}\right) = \frac{119}{3}$

(ii) $\alpha^2 + \beta^2 + \gamma^2 = (\alpha + \beta + \gamma)^2 - 2(\alpha\beta + \alpha\gamma + \beta\gamma)$

$= \left(-\frac{7}{3}\right)^2 - 2\left(\frac{11}{3}\right)$

$= -\frac{17}{9}$

(iii) By the complex conjugate theorem, $P(x)$ either has two or no complex roots.

Since $\alpha^2 + \beta^2 + \gamma^2 < 0$, then the roots of $P(x)$ can't all be real $\therefore$ $P(x)$ has 1 real root and 2 complex roots.

(b) (i) $\angle ALH = \angle DCF = \alpha$ ($\angle$'s in the same segment)

$\angle DAC = 90^\circ - \alpha$ ($\angle$ sum of Δ)

$\angle AFK = 90^\circ$

$\therefore \angle AHF = \alpha$ ($\angle$ sum of Δ)

(ii) ΔAHL is isos. (base $\angle$'s equal)

$\therefore AH = AL$ (equal sides of isos. Δ)

(iii) $\angle AMH = \angle ABD = \beta$ ($\angle$'s in the same segment)

$\angle MAF = 90^\circ - \beta$

$\angle BAH = 90^\circ - \beta$

$\angle FHA = \beta$ ($\angle$ sum of Δ)

$\therefore AM = AH$ (base $\angle$'s of isos. Δ)

(iv) $\angle MAL = 2\times(90^\circ - \beta) + 2\times(90^\circ - \alpha)$

$\angle BAC = (90^\circ - \beta) + (90^\circ - \alpha)$

$\therefore$ Arc $BKC = \frac{1}{2}$ Arc MCL

(c) (i) $\frac{xx_0}{a^2} + \frac{yy_0}{b^2} = 1$

Substituting $S(ae, 0)$ gives:

$\frac{aex_0}{a^2} = 1$

$x_0 = \frac{a}{e}$

$\therefore T$ lies on the directrix.

(ii) $ST = \left(\frac{a}{e} - ae\right)$

By the focus-directrix property of the ellipse:

$PS = e\,PM$

where PM is the perpendicular distance to the directrix.

Let $P(x_1, y_1)$ and $T\left(\frac{a}{e}, 0\right)$

$\therefore \frac{x_1 x_0}{a^2} = 1$

$x_1 = \frac{a^2}{x_0} = \frac{a^2}{\frac{a}{e}} = ae$

$\therefore PS = e\left(\frac{a}{e} - x_1\right) = e\left(\frac{a}{e} - ae\right)$

Thus $\frac{PS}{ST} = e$

(iii) Let $\angle PTS = \theta$

$\therefore \tan\theta = \frac{PS}{PT} = e < 1$ i.e. $\theta < 45^\circ$

$\therefore \angle PTQ = 2\theta < 90^\circ$.

(iv) From (iii) point P, S and Q are collinear.

Thus area of ΔPQT is given by:

$PS \times ST = e\left(\frac{a}{e} - ae\right) \times \left(\frac{a}{e} - ae\right)$

$= (a - ae^2)\left(\frac{a}{e} - ae\right)$

$= a^2(1 - e^2)\left(\frac{1}{e} - e\right)$

$= b^2\left(\frac{1}{e} - e\right)$ units^2

Question 5

(i) $ax = x^2 - ax$

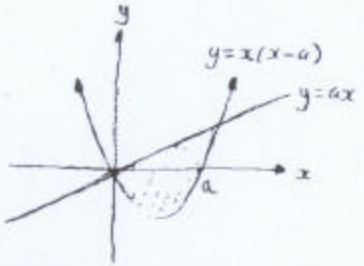
$x^2 - 2ax = 0$

$x(x-2a) = 0$

$x = 0, 2a$

Points of intersection are $(0,0)$ and $(2a, 2a^2)$

(ii)



(iii) $V = \int_0^{2a} 2\pi \left(\begin{smallmatrix}\text{shell}\\ \text{radius}\end{smallmatrix}\right)\left(\begin{smallmatrix}\text{shell}\\ \text{height}\end{smallmatrix}\right) dx$

$= 2\pi \int_0^{2a} (2a+x)(ax - (x^2 - ax))\, dx$

$= 2\pi \int_0^{2a} (2a+x)(2ax - x^2)\, dx$

$= 2\pi \int_0^{2a} 4a^2x - 2ax^2 + 2ax^2 - x^3\, dx$

$= 2\pi \left[2a^2x^2 - \frac{x^4}{4}\right]_0^{2a} = 2\pi\left[8a^4 - 4a^4\right]$

$= 8\pi a^4 \text{ units}^3$

(b) (i) $(2^n - 2)$ ways

(ii) $3^5 - \left(3 + (2^5 - 2) \times {}^3C_2\right) = 150$ ways

(c) (i) The forces acting on P are:

- A centripetal force directed towards the centre: $mr\omega^2$
- Its weight mg
- The normal reaction N

Resolving forces at P vertically:

$mg = N\cos\theta$ ..... (1)

Resolving forces at P horizontally:

$mr\omega^2 = N\sin\theta$ ..... (2)

(ii) From (1) $m = \frac{N\cos\theta}{g}$

i.e. $\frac{N\cos\theta}{g} \cdot r\omega^2 = N\sin\theta$

$\cos\theta = \frac{g}{r\omega^2} \cdot \sin\theta$

$= \frac{g}{r\omega^2} \cdot \frac{r}{R} \quad \left(\sin\theta = \frac{r}{R}\right)$

$= \frac{g}{R\omega^2}$

Hence for the marble to remain in contact with the surface of the sphere then $\theta = 0$ or $\cos\theta = \frac{g}{R\omega^2}$. Note for $\theta = 0$ the marble rests at the bottom and so remains in contact with the sphere.

(iii) $\cos\theta < 1$, for $0 < \theta < 90°$

i.e. $\frac{g}{R\omega^2} < 1$

$R\omega^2 > g$

$\omega^2 > \frac{g}{R}$

$\omega > \sqrt{\frac{g}{R}}$

Question 6

(a) (i) $\int_0^{\pi} \frac{\sin x}{1+\cos^2 x}\, dx$ Let $u = \cos x$

$\frac{du}{dx} = -\sin x$

$-du = \sin x\, dx$

$= \int_1^{-1} \frac{-du}{1+u^2}$

$= \int_{-1}^{1} \frac{1}{1+u^2}\, du = \left[\tan^{-1} u\right]_{-1}^{1} = \frac{\pi}{4} - \left(-\frac{\pi}{4}\right) = \frac{\pi}{2}$

(ii) $\int_0^{\pi} \frac{x\sin x}{1+\cos^2 x}\, dx$ Let $x = \pi - u$

$dx = -du$

$\int_0^{\pi} \frac{x\sin x}{1+\cos^2 x}\, dx = \int_{\pi}^{0} \frac{(\pi-u)\sin(\pi-u)}{1+\cos^2(\pi-u)} \cdot -du$

$I = \int_0^{\pi} \frac{\pi\sin u}{1+\cos^2 u} - \int_0^{\pi} \frac{u\sin u}{1+\cos^2 u}$

$2I = \pi \times \frac{\pi}{2}$

$I = \frac{\pi^2}{4}$

(b) (i) $\ddot{x} = g - kv$

$\frac{dv}{dt} = g - kv$

$\frac{dt}{dv} = \frac{1}{g-kv}$

i.e. $t = \int \frac{1}{g-kv}\, dv = -\frac{1}{k}\ln(g-kv) + c$

when $t = 0$, $v = A$ $\therefore c = \frac{1}{k}\ln(g-kA)$

$\therefore t = \frac{1}{k}\ln\left(\frac{g-kA}{g-kv}\right)$

$e^{kt} = \frac{g-kA}{g-kv}$

$ge - kve^{kt} = g - kA$

$kve^{kt} = ge^{kt} - g + kA$

$v = \frac{g}{k} - \frac{g}{k}e^{-kt} + Ae^{-kt}$

as $t \to \infty$, $v \to B$

$\therefore B = \frac{g}{k} \to k = \frac{g}{B}$

$\frac{d}{dt}(ve^{kt}) = \frac{d}{dt}\left(\frac{g}{k}e^{kt} - \frac{g}{k} + A\right)$

$= \frac{g}{k}e^{kt} \cdot k$

$= ge^{kt}$

$\frac{d}{dt}(ve^{kt}) = ge^{kt}$

Integrating both sides w.r.t. t:

$ve^{kt} = \frac{g}{k}e^{kt} + C$

when $t = 0$, $v = A$ $\therefore C = A - \frac{g}{k}$

$ve^{kt} = \frac{g}{k}e^{kt} + A - \frac{g}{k}$

$ve^{kt} = Be^{kt} + A - B$

$v = B - (B-A)e^{-kt} = B - (B-A)e^{-\frac{gt}{B}}$

$\frac{dx}{dt} = B - (B-A)e^{-\frac{gt}{B}}$

Integrating both sides w.r.t. t:

$x = Bt + (B-A)e^{-\frac{gt}{B}} \times \frac{B}{g} + C$

when $t = 0$, $x = 0$ $\therefore C = -\frac{B}{g}(B-A)$

$\therefore x = Bt - \frac{B}{g}(B-A)\left[1 - e^{-\frac{gt}{B}}\right]$

(iv) Equating displacement for $t = T$, gives:

$BT - \frac{B}{g}(B-A)\left[1 - e^{-\frac{gT}{B}}\right] = h + BT - \frac{B}{g}(B+A)\left[1 - e^{-\frac{gT}{B}}\right]$

$-\frac{B}{g}(B-A) + \frac{B}{g}(B-A)e^{-\frac{gT}{B}} = h - \frac{B}{g}(B+A) + \frac{B}{g}(B+A)e^{-\frac{gT}{B}}$

$\frac{2AB}{g} - h = \frac{2AB}{g}e^{-\frac{gT}{B}}$

$2AB - gh = 2ABe^{-\frac{gT}{B}}$

$e^{\frac{gT}{B}} = \frac{2AB}{2AB - gh}$

$T = \frac{B}{g}\log_e\left(\frac{2AB}{2AB - gh}\right)$

(vi) $2AB - gh > 0$

$2AB > gh$

$A > \frac{gh}{2B}$

Question 7

(a) (i) $(a-1)^2 \geq 0$

$a^2 - 2a + 1 \geq 0$

$a - 2 + \frac{1}{a} \geq 0$

$a + \frac{1}{a} \geq 2$

(ii) Step 1: Test result for $n = 1$

$LHS = a_1 \times \frac{1}{a_1} = 1$

$RHS = n^2 = 1^2 = 1$

Thus $LHS \geq 1$ $\therefore$ result hold for $n = 1$

Step 2: Assume the result true for $n = k$

$(a_1 + a_2 + \ldots + a_k)\left(\frac{1}{a_1} + \frac{1}{a_2} + \ldots + \frac{1}{a_k}\right) \geq k^2$

Now let's prove it for $n = k+1$. We have

$(a_1 + a_2 + \ldots + a_k + a_{k+1})\left(\frac{1}{a_1} + \frac{1}{a_2} + \ldots + \frac{1}{a_k} + \frac{1}{a_{k+1}}\right)$

$= (a_1 + a_2 + \ldots + a_k)\left(\frac{1}{a_1} + \frac{1}{a_2} + \ldots + \frac{1}{a_k}\right)$

$+ a_{k+1}\left(\frac{1}{a_1} + \ldots + \frac{1}{a_k}\right) + \frac{1}{a_{k+1}}(a_1 + \ldots + a_k) + 1$

$\geq k^2 + \sum_{i=1}^{k}\left(\frac{a_{k+1}}{a_i} + \frac{a_i}{a_{k+1}}\right) + 1$

[Note: The sum of k positive no.'s and their reciprocals is at least $2k$ from (i).]

$\geq k^2 + 2k + 1 = (k+1)^2$

This proves the inequality for $n = k+1$ and so on by the axiom of induction, the result is true for all $n \in \mathbb{N}$

(iii) $(\sin^2\theta + \cos^2\theta + \tan^2\theta)\left(\frac{1}{\sin^2\theta} + \frac{1}{\cos^2\theta} + \frac{1}{\tan^2\theta}\right) \geq$

$(1 + \tan^2\theta)(\cos ec^2\theta + \sec^2\theta + \cot^2\theta) \geq 9$

$\sec^2\theta(\cos ec^2\theta + \sec^2\theta + \cot^2\theta) \geq 9$

$\cos ec^2\theta + \sec^2\theta + \cot^2\theta \geq 9\cos^2\theta$

(b) (i) $z + \frac{1}{2} = 2\cos\alpha$

$z^2 + 1 = 2z\cos\alpha$

$z^2 - (2\cos\alpha)z + 1 = 0$

$z = \frac{2\cos\alpha \pm \sqrt{4\cos^2\alpha - 4}}{2}$

$= \frac{2\cos\alpha \pm 2\sqrt{\cos^2\alpha - 1}}{2}$

$= \cos\alpha \pm i\sin\alpha$

$z^n = \cos n\alpha + i\sin n\alpha$ by De Moivre's theorem

$z^{-n} = \cos(-n\alpha) + i\sin(-n\alpha) = \cos n\alpha - i\sin n\alpha$

$\therefore z^n + \frac{1}{z^n} = 2\cos n\alpha$.

(ii) $w^3 = \left(z + \frac{1}{z}\right)^3 = z^3 + 3z + \frac{3}{z} + \frac{1}{z^3}$

$w^3 = z^3 + \frac{1}{z^3} + 2$

$w^3 + w^2 - 2w - 2 = z^3 + 3z + \frac{3}{z} + \frac{1}{z^3} + z^2 + \frac{1}{z^2} + 2 - 2z - \frac{2}{z}$

$= \left(z^3 + \frac{1}{z^3}\right) + \left(z^2 + \frac{1}{z^2}\right) + \left(z + \frac{1}{z}\right) - 2$

From (ii):

$\cos x + \cos 2x + \cos 3x = 0$ (dividing by 2)

$\therefore \; w^3 + w^2 - 2w - 2 = 0$

$w^2(w+1) - 2(w+1) = 0$

$(w+1)(w^2 - 2) = 0$

$w = -1$ or $w = \pm\sqrt{2}$

$2\cos x = -1$ $\qquad$ $2\cos x = \pm\sqrt{2}$

$\cos x = -\frac{1}{2}$ $\qquad$ $\cos x = \pm\frac{1}{\sqrt{2}}$

$x = \frac{2\pi}{3}, \frac{4\pi}{3}$ $\qquad$ $x = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$

Question 8

Area of $\Delta OPP' = \frac{1}{2} \times OP' \times PP'$

$= \frac{1}{2} \times P \times \frac{1}{P} = \frac{1}{2}$ units^2

To prove the areas of OPQ and $Q'QPP'$ are the same all we need to do is show that the areas of $\Delta OPP'$ and $\Delta OQQ'$ are equal.

Area of $\Delta OQQ' = \frac{1}{2} \times q \times \frac{1}{q} = \frac{1}{2}$ units^2 = (i)

$\therefore$ Area of OPQ = Area of $Q'QPP'$.

(ii) Let M' be the point on x-axis directly below M.

$\Delta ORR' \;|||\; \Delta OMM'$ (AAA)

i.e. $\frac{MM'}{OM'} = \frac{RR'}{OR'}$

$M\left(\frac{p+q}{2}, \frac{p+q}{2pq}\right)$

$\therefore \; \frac{MM'}{OM'} = \frac{p+q}{2pq} \times \frac{2}{p+q} = \frac{1}{pq}$

$\frac{RR'}{OR'} = \frac{\frac{1}{r}}{r} = \frac{1}{r^2}$

$\therefore \; \frac{1}{pq} = \frac{1}{r^2} \Rightarrow r^2 = pq$

(iii) Area of region $R'RPP' = \int_{\frac{1}{r}}^{\frac{1}{p}} \frac{1}{x}\,dx$

$= \left[\ln x\right]_{\frac{1}{r}}^{\frac{1}{p}}$

$= \ln\left(\frac{r}{p}\right) \qquad r = \sqrt{pq}$

$= \ln\left(\sqrt{\frac{q}{p}}\right)$

Area of region $Q'QRR' = \int_{\frac{1}{q}}^{\frac{1}{r}} \frac{1}{x}\,dx$

$= \left[\ln x\right]_{\frac{1}{q}}^{\frac{1}{r}}$

$= \ln\left(\frac{q}{r}\right)$

$= \ln\left(\frac{q}{\sqrt{pq}}\right)$

$= \ln\left(\sqrt{\frac{q}{p}}\right)$

= Area of region $RR'PP'$

(iv) Since M is the midpoint of PQ

$\therefore \; PM = MQ$

Areas of ΔOPM and ΔOMQ are equal as they have the same height and base. Also since PMR and MRQ are equivalent by symmetry, thus OR divides the shaded region OPQ into 2 equal areas.

(v) $I_n + I_{n+2} = \int_0^{\frac{\pi}{4}} \tan^n x\,dx + \int_0^{\frac{\pi}{4}} \tan^n x \cdot \tan^2 x\,dx$

$= \int_0^{\frac{\pi}{4}} \tan^n x\,dx + \int_0^{\frac{\pi}{4}} \tan^n x \cdot (\sec^2 x - 1)$

$= \int_0^{\frac{\pi}{4}} \tan^n x \cdot \sec^2 x\,dx$

$= \left[\frac{\tan^{n+1} x}{n+1}\right]_0^{\pi/4} = \frac{1}{n+1}$

(ii) $I_n - I_{n-1} = (-1)^n I_{2n} - (-1)^{n-1} I_{2n-2}$

$= (-1)^n \left[I_{2n-2} + I_{(2n-2)+2} \right.$

$= (-1)^n \cdot \frac{1}{2n-2+1} = \frac{(-1)^n}{2n-1}, \quad n \geq$

(iii) $I_m = (I_m - I_{m-1}) + (I_{m-1} - I_{m-2}) +$

$\ldots + (I_1 - I_0) + I_0$

$= I_0 + \sum_{n=1}^{m} (I_n - I_{n-1})$

$= (-1)^0 I_{2(0)} + \sum_{n=1}^{m} \frac{(-1)^n}{2n-1}$

$= I_0 + \sum_{n=1}^{m} \frac{(-1)^n}{2n-1}$

$= \frac{\pi}{4} + \sum_{n=1}^{m} \frac{(-1)^n}{2n-1}$

(iv) $I_n = \int_0^{\frac{\pi}{4}} \tan^n x\,dx$ $\qquad u = \tan x$

$\frac{du}{dx} = \sec^2 x$

$= \int_0^1 \frac{u^n}{(1+u^2)}\,du$ $\qquad du = (1 + \tan^2 x)\,dx$

$du = (1 + u^2)\,dx$

(v) $I_n + I_{n+2} = \frac{1}{n+1}$, since I_{n+2} is positive and I_n is also positive $\therefore \; 0 \leq I_n \leq \frac{1}{n+1}$.

As $n \to \infty$, $\frac{1}{n+1} \to 0$

$\therefore \; I_n \to 0$, $I_{2n} \to 0$ and

$J_n = (-1)^n I_{2n} \to 0$.