Solution to Maths Extension 2 HSC 2003 (written by Terry Lee)

QUESTION 1

a) $\displaystyle\int_0^1 \frac{e^x}{(1+e^x)^2}dx = \left[\frac{-1}{1+e^x}\right]_0^1 = \frac{1}{2} - \frac{1}{1+e}$

b) Let $u = \ln x, du = \dfrac{1}{x}dx$

Let $dv = x^3dx, v = \dfrac{x^4}{4}$

$\therefore \displaystyle\int x^3 \ln x dx = \frac{x^4}{4}\ln x - \frac{1}{4}\int x^3 dx$

$= \dfrac{x^4}{4}\ln x - \dfrac{x^4}{16} + C$

c) $\displaystyle\int \frac{dx}{\sqrt{x^2-2x+5}} = \int \frac{dx}{\sqrt{(x-1)^2+4}}$

$= \ln(x-1+\sqrt{x^2-2x+5}) + C$

d) i) Put $x = 1$ gives $a = \dfrac{5-3+13}{5} = 3,$

Equating the coefficients of x^2 gives

$5 = 3 + b, \therefore b = 2$

ii) $\displaystyle\int \left(\frac{3}{x-1} + \frac{2x-1}{x^2+4}\right)dx$

$= 3\ln(x-1) + \ln(x^2+4) - \dfrac{1}{2}\tan^{-1}\dfrac{x}{2} + C$

e) Let $x = 3\sin\theta, dx = 3\cos\theta\, d\theta.$

When $x = 0, \theta = 0;$ when $x = \dfrac{3}{\sqrt{2}}, \theta = \dfrac{\pi}{4}$

$\displaystyle\int_0^{\pi/4} \frac{3\cos\theta d\theta}{27\cos^3\theta} = \frac{1}{9}\int_0^{\pi/4} \sec^2\theta d\theta$

$= \dfrac{1}{9}\left[\tan\theta\right]_0^{\pi/4}$

$= \dfrac{1}{9}\left(\tan\dfrac{\pi}{4} - \tan 0\right)$

$= \dfrac{1}{9}$

QUESTION 2

a) i) $z\overline{w} = (2+i)(1+i) = 1+3i$

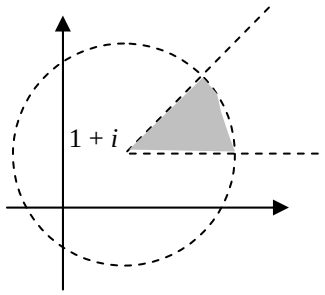
ii) $\dfrac{4}{z} = \dfrac{4}{2+i} = \dfrac{4(2-i)}{5}$

b) i) $\alpha = -1+i = \sqrt{2}\,\text{cis}\,\dfrac{3\pi}{4}$

ii) $\alpha^4 = (\sqrt{2})^4\,\text{cis}\,3\pi = -4$

iii) $-1 + i$ and $-1 - i$ are the roots (since the coefficients are real), $\therefore$ The sum of these roots = -2, and their product = 2, $\therefore$ The factor is $z^2 + 2z + 2$

c)



d) $(\cos\theta + i\sin\theta)^5 = \cos 5\theta + i\sin 5\theta$

$= c^5 + 5c^4(is) + 10c^3(is)^2 + 10c^2(is)^3$

$+5c(is)^4 + (is)^5$

$= \left(c^5 - 10c^3s^2 + 5cs^4\right)$

$+i\left(5c^4s - 10c^2s^3 + s^5\right)$

where $c = \cos\theta$ and $s = \sin\theta$

$\therefore \cos 5\theta = \cos^5\theta - 10\cos^3\theta(\sin^2\theta)$

$+5\cos\theta(\sin^2\theta)^2$

$= \cos^5\theta - 10\cos^3\theta(1-\cos^2\theta)$

$+5\cos\theta(1-\cos^2\theta)^2$

$= \cos^5\theta - 10\cos^3\theta(1-\cos^2\theta)$

$+5\cos\theta(1-2\cos^2\theta+\cos^4\theta)$

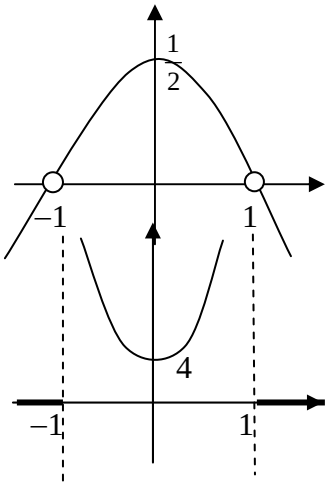
$= 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta$

QUESTION 3

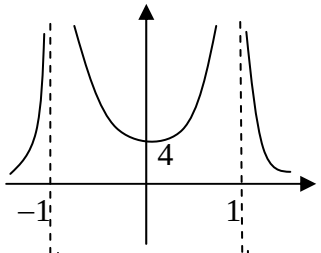
a)

i)

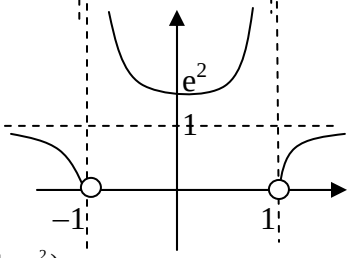
iii)



iii)



iv)



b) $4 = 9(1-e^2)$,

$e^2 = 1-\frac{4}{9} = \frac{5}{9}$

$\therefore e = \frac{\sqrt{5}}{3}$

$\therefore$ Foci $(\pm\sqrt{5}, 0)$

Directrices: $x = \pm\frac{9}{\sqrt{5}}$

c) i) On the annulus let (x, y), $x < 2$, be a point of the curve, so, the outer radius of the annulus is $3 - x$. As half of the line segment ℓ is $2 - x$, the inner radius is $3-(x+2(2-x)) = x-1$.

$\therefore$ Area of the annulus = $\pi(3-x)^2 - \pi(x-1)^2$

$= \pi(9-6x+x^2) - \pi(x^2-2x+1)$

$= \pi(8-4x)$

$= 4\pi(2-x)$ (1)

$y = (x-1)(3-x) = -3+4x-x^2$

$= 1-(x-2)^2$

$\therefore (x-2)^2 = 1-y$

$(x-2) = -\sqrt{1-y}$, since $x < 2, \therefore x-2 < 0$

$\therefore$ From (1), Area $A = 4\pi\sqrt{1-y}$

Alternatively,

Let x_1 and x_2 be the two solutions of $(x-1)(3-x) = y$,

$x_1 < x_2$.

$-x^2+4x-3 = y$

$x^2-4x+3 = -y$

$(x-2)^2 = 1-y$

$x-2 = \pm\sqrt{1-y}$

$\therefore x = 2 \pm \sqrt{1-y}$

$\therefore x_1 = 2-\sqrt{1-y}$ and $x_2 = 2+\sqrt{1-y}$

Area of annulus = $\pi\left((3-x_1)^2 - (3-x_2)^2\right)$

$= \pi\left(\left(3-(2-\sqrt{1-y})\right)^2 - \left(3-(2+\sqrt{1-y})\right)^2\right)$

$= \pi\left(\left(1+\sqrt{1-y}\right)^2 - \left(1-\sqrt{1-y}\right)^2\right)$

$= 4\pi\sqrt{1-y}$

Alternatively,

Let x_1 and x_2 be the two solutions of $(x-1)(3-x) = y$,

$x_1 < x_2$.

$x^2-4x+3+y = 0$

$\therefore x_1 + x_2 = 4,$

$x_2 - x_1 = \sqrt{(x_1+x_2)^2 - 4x_1x_2}$

$= \sqrt{16-4(3+y)}$

$= 2\sqrt{1-y}$

$\therefore$ Area = $\pi\left((3-x_1)^2 - (3-x_2)^2\right)$

$= \pi\left(6(x_2-x_1) + (x_1^2 - x_2^2)\right)$

$= \pi\left(6(x_2-x_1) - (x_2-x_1)(x_1+x_2)\right)$

$= \pi\left(6\times 2\sqrt{1-y} - 2\sqrt{1-y}\times 4\right)$

$= \pi\left(12\sqrt{1-y} - 8\sqrt{1-y}\right)$

$= 4\pi\sqrt{1-y}$

ii) The volume of the solid:

$V = 4\pi\int_0^1 \sqrt{1-y}\,dy,$ noting $x = 1, y = 0$

and $x = 2, y = 1$

$= 4\pi\left[\frac{-2(1-y)^{3/2}}{3}\right]_0^1$

$= \frac{8\pi}{3}$

QUESTION 4

a) i) Noting $\omega = \frac{d\theta}{dt}$ and $\frac{d\omega}{dt} = 0$,

$x = r\cos\theta, \dot{x} = -r\sin\theta.\omega, \ddot{x} = -r\cos\theta.\omega^2$

$y = r\sin\theta, \dot{y} = r\cos\theta.\omega, \ddot{y} = -r\sin\theta.\omega^2$

$\therefore$ The two components of the force are

$F_x = -mr\cos\theta.\omega^2$ and $F_y = -mr\sin\theta.\omega^2$

$\therefore$ Its magnitude = $\sqrt{F_x^2 + F_y^2}$

$= mr\omega^2\sqrt{\cos^2\theta + \sin^2\theta}$

$= mr\omega^2$

and its inclination angle $= \tan^{-1}\dfrac{-mr\sin\theta.\omega^2}{-mr\cos\theta.\omega^2}$

$= \tan^{-1}(\tan\theta) = \theta$

Since both components head in the negative direction, this is an inward force.

ii) $\dfrac{Am}{r^2} = mr\omega^2$

$r^3 = \dfrac{A}{\omega^2}$

$\therefore r = \sqrt[3]{\dfrac{A}{\omega^2}}$

b) i) $\dfrac{dy}{dx} = \dfrac{dy/d\theta}{dx/d\theta} = \dfrac{b\sec^2\theta}{a\sec\theta\tan\theta} = \dfrac{b\sec\theta}{a\tan\theta}$

The equation of the tangent is

$y - b\tan\theta = \dfrac{b\sec\theta}{a\tan\theta}(x - a\sec\theta)$

$a\tan\theta y - ab\tan^2\theta = b\sec\theta x - ab\sec^2\theta$

$b\sec\theta x - a\tan\theta y = ab\left(\sec^2\theta - \tan^2\theta\right)$

$b\sec\theta x - a\tan\theta y = ab \qquad (5)$

c) ii) Put $y = \dfrac{bx}{a}$ in (5) gives

$b\sec\theta x - a\tan\theta\left(\dfrac{bx}{a}\right) = ab$

$bx(\sec\theta - \tan\theta) = ab$

$\therefore x = \dfrac{a}{\sec\theta - \tan\theta}$

$= \dfrac{a}{\dfrac{1}{\cos\theta} - \dfrac{\sin\theta}{\cos\theta}}$

$= \dfrac{a\cos\theta}{1-\sin\theta}$

Put $x = \dfrac{a\cos\theta}{1-\sin\theta}$ in $y = \dfrac{bx}{a}$ gives $y = \dfrac{b\cos\theta}{1-\sin\theta}$

$\therefore A\left(\dfrac{a\cos\theta}{1-\sin\theta}, \dfrac{b\cos\theta}{1-\sin\theta}\right)$

Similarly, put $y = -\dfrac{bx}{a}$ in (5) gives

$x = \dfrac{a\cos\theta}{1+\sin\theta}, y = -\dfrac{b\cos\theta}{1+\sin\theta}$

$\therefore B\left(\dfrac{a\cos\theta}{1+\sin\theta}, \dfrac{-b\cos\theta}{1+\sin\theta}\right)$

iii) The area of OAB =

$= \dfrac{1}{2}OA \times OB \times \sin\angle AOB,$ where

$OA = \sqrt{\left(\dfrac{a\cos\theta}{1-\sin\theta}\right)^2 + \left(\dfrac{b\cos\theta}{1-\sin\theta}\right)^2}$

$= \sqrt{a^2+b^2}\left(\dfrac{\cos\theta}{1-\sin\theta}\right)$

$OB = \sqrt{\left(\dfrac{a\cos\theta}{1+\sin\theta}\right)^2 + \left(\dfrac{b\cos\theta}{1+\sin\theta}\right)^2}$

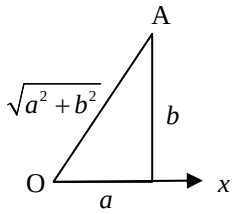
$= \sqrt{a^2+b^2}\left(\dfrac{\cos\theta}{1+\sin\theta}\right)$

$\sin\angle AOB = 2\sin\angle AOx\cos\angle AOx$

$= 2\dfrac{b}{\sqrt{a^2+b^2}}\dfrac{a}{\sqrt{a^2+b^2}}$, noting $\tan\angle AOx = \dfrac{b}{a}$

$\therefore A = \dfrac{1}{2}\sqrt{a^2+b^2}\left(\dfrac{\cos\theta}{1-\sin\theta}\right) \times \sqrt{a^2+b^2}\left(\dfrac{\cos\theta}{1+\sin\theta}\right) \times 2\dfrac{b}{\sqrt{a^2+b^2}}\dfrac{a}{\sqrt{a^2+b^2}}$

$= ab$



c)

i) n^n

ii) P(at least 1 door not chosen)

= 1 – P(each door is chosen once)

$= 1 - \dfrac{n!}{n^n}$

QUESTION 5

a)

i) $s_1 = \alpha + \beta + \gamma = -\dfrac{b}{a} = 0$

$s_2 = \alpha^2 + \beta^2 + \gamma^2 = (\alpha+\beta+\gamma)^2 - 2(\alpha\beta + \alpha\gamma + \beta\gamma)$

$= -2p$

$s_3 = \alpha^3 + \beta^3 + \gamma^3 = -p(\alpha+\beta+\gamma) - 3q,$

using $x^3 = -px - q,$ hence $\alpha^3 = p\alpha - q$

$\therefore s_3 = -3q$

ii) From $x^3 = -px - q$

$x^n = -px^{n-2} - qx^{n-3}$, on multiplying by x^{n-3}

$\therefore \alpha^n = -p\alpha^{n-2} - q\alpha^{n-3}$

$\beta^n = -p\beta^{n-2} - q\beta^{n-3}$

$\gamma^n = -p\gamma^{n-2} - q\gamma^{n-3}$

Adding the above three lines,

$s_n = -ps_{n-2} - qs_{n-3}$

iii) $\alpha^5 + \beta^5 + \gamma^5 = -p(\alpha^3 + \beta^3 + \gamma^3) - q(\alpha^2 + \beta^2 + \gamma^2)$

From part (i), $-p = \dfrac{\alpha^2 + \beta^2 + \gamma^2}{2}, -q = \dfrac{\alpha^3 + \beta^3 + \gamma^3}{3}$

$$\therefore \alpha^5 + \beta^5 + \gamma^5 = \frac{\alpha^2 + \beta^2 + \gamma^2}{2}(\alpha^3 + \beta^3 + \gamma^3) + \frac{\alpha^3 + \beta^3 + \gamma^3}{3}(\alpha^2 + \beta^2 + \gamma^2)$$

$$= (\alpha^3 + \beta^3 + \gamma^3)\left(\frac{5}{6}(\alpha^2 + \beta^2 + \gamma^2)\right)$$

$$\therefore \frac{\alpha^5 + \beta^5 + \gamma^5}{5} = \frac{(\alpha^3 + \beta^3 + \gamma^3)(\alpha^2 + \beta^2 + \gamma^2)}{6}$$

b)

i) $\ddot{x} = \dfrac{d\dot{x}}{dt} = -k\dot{x}$

$\dfrac{dt}{d\dot{x}} = -\dfrac{1}{k\dot{x}}$

$t = -\dfrac{1}{k}\ln \dot{x} + C$

When $t = 0, \dot{x} = u\cos\alpha$,

$0 = -\dfrac{1}{k}\ln(u\cos\alpha) + C, \therefore C = \dfrac{1}{k}\ln(u\cos\alpha)$

$\therefore t = \dfrac{1}{k}\ln\dfrac{u\cos\alpha}{\dot{x}}$

$\dfrac{u\cos\alpha}{\dot{x}} = e^{kt}$

$\therefore \dot{x} = \dfrac{u\cos\alpha}{e^{kt}}$

$= ue^{-kt}\cos\alpha$

ii) Differentiating $\dot{y} = \dfrac{1}{k}\left((ku\sin\alpha + g)e^{-kt} - g\right)$ with respect to time gives

$\dfrac{d\dot{y}}{dt} = \ddot{y} = -(ku\sin\alpha + g)e^{-kt}$

$= -(k\dot{y} + g)$

Noting $\dot{y} = \dfrac{1}{k}\left((ku\sin\alpha + g)e^{-kt} - g\right)$, so

$(ku\sin\alpha + g)e^{-kt} = k\dot{y} + g$

Further, when $t = 0, \dot{y} = \dfrac{1}{k}\left((ku\sin\alpha + g) - g\right)$

$= u\sin\alpha$, which is true.

$\therefore \dot{y} = \dfrac{1}{k}\left((ku\sin\alpha + g)e^{-kt} - g\right)$ satisfies the equation of the vertical motion and initial condition.

iii) Maximum height is reached when $\dot{y} = 0$,

$\therefore e^{-kt} = \dfrac{g}{ku\sin\alpha + g}$

$\therefore t = -\dfrac{1}{k}\ln\dfrac{g}{ku\sin\alpha + g}$

$= \dfrac{1}{k}\ln\left(1 + \dfrac{ku\sin\alpha}{g}\right)$

iv) When $t \to \infty, \dot{x} = u\cos\alpha.e^{-kt} \to 0$

$\ddot{x} = \dot{x}\dfrac{d\dot{x}}{dx} = -k\dot{x}$

$\therefore \dfrac{d\dot{x}}{dx} = -k$

$\therefore \dot{x} = -kx + C$

When $x = 0, \dot{x} = u\cos\alpha, \therefore C = u\cos\alpha$

$\therefore \dot{x} = -kx + u\cos\alpha$

$\therefore$ When $t \to \infty, \dot{x} \to 0, x \to \dfrac{u\cos\alpha}{k}$

QUESTION 6

a)

i) LHS = $\cos ax\cos bx - \sin ax\sin bx + \cos ax\cos bx + \sin ax\sin bx$

$= 2\cos ax\cos bx$ = RHS

ii) $\int \cos 3x\cos 2x dx = \dfrac{1}{2}\int(\cos 5x + \cos x)dx$

$= \dfrac{\sin 5x}{10} + \dfrac{\sin x}{2} + C$

b)

i) $s_3 = s_2 + 2s_1 = 2 + 2 = 4$

$s_4 = s_3 + 3s_2 = 4 + 6 = 10$

ii) $\left(x + \sqrt{x}\right)^2 = x^2 + 2x\sqrt{x} + x \geq x^2 + x$ if $x \geq 0$

$\therefore x + \sqrt{x} \geq \sqrt{x^2 + x}$

iii) $s_1 = 1 \geq \sqrt{1!}$ and $s_2 = 2 \geq \sqrt{2!}$

Assume $s_k \geq \sqrt{k!}$ and $s_{k-1} \geq \sqrt{(k-1)!}$

Required to prove $s_{k+1} \geq \sqrt{(k+1)!}$

$s_{k+1} = s_k + ks_{k-1}$, by definition

$\geq \sqrt{k!} + k\sqrt{(k-1)!}$, by asumption

$\geq \sqrt{k}\sqrt{(k-1)!} + k\sqrt{(k-1)!}$

$\geq \sqrt{(k-1)!}\left(\sqrt{k} + k\right)$

$\geq \sqrt{(k-1)!}\sqrt{k(k+1)}$, from part (ii)

$\geq \sqrt{(k+1)!}$

$\therefore s_n \geq \sqrt{n!}$ for all $n \geq 1$

c)

i) $\left(\sqrt{x} - \sqrt{y}\right)^2 \geq 0$

$x + y - 2\sqrt{xy} \geq 0$

$\frac{x+y}{2} \geq \sqrt{xy}$

ii) $a^4 + b^4 \geq 2a^2b^2$

$a^4 + c^4 \geq 2a^2c^2$

$b^4 + c^4 \geq 2b^2c^2$

Adding the three lines above,

$2(a^4 + b^4 + c^4) \geq 2(a^2b^2 + a^2c^2 + b^2c^2)$

$\therefore a^4 + b^4 + c^4 \geq a^2b^2 + a^2c^2 + b^2c^2$

iii) $a^2b^2 + a^2c^2 \geq 2ab.ac = 2a^2bc$

$a^2b^2 + b^2c^2 \geq 2ab.bc = 2ab^2c$

$a^2c^2 + b^2c^2 \geq 2ac.bc = 2abc^2$

Adding the three lines above,

$2(a^2b^2 + a^2c^2 + b^2c^2) \geq 2(a^2bc + ab^2c + abc^2)$

$a^2b^2 + a^2c^2 + b^2c^2 \geq a^2bc + ab^2c + abc^2$

iv) $a^4 + b^4 + c^4 \geq a^2b^2 + a^2c^2 + b^2c^2$

$\geq a^2bc + ab^2c + abc^2$

$\geq abc(a + b + c)$

$\geq abcd$

QUESTION 7

a) Consider a strip of thickness δx, height y, that cuts the curve at the point (x,y). As this strip is rotated about the y-axis, it becomes a hollow cylinder of volume

$\delta V = \pi\left((x + \delta x)^2 - x^2\right)y$

$= 2\pi xy\delta x$, noting $(\delta x)^2 \ll (\delta x)$

$\therefore$ The volume of the solid

$V = 2\pi\int_0^{\sqrt{3}} xydx$

$= 2\pi\int_0^{\sqrt{3}} x^2(3 - x^2)dx$

$= 2\pi\int_0^{\sqrt{3}} \left(3x^2 - x^4\right)dx$

$= 2\pi\left[x^3 - \frac{x^5}{5}\right]_0^{\sqrt{3}}$

$= 2\pi\left[3\sqrt{3} - \frac{9\sqrt{3}}{5}\right]$

$= \frac{12\sqrt{3}\pi}{5}$

b)

i) $\angle P$ is common

$\angle PSB = \angle SAP$ (angle in alternate segment)

$\therefore \Delta ASP \,|||\, \Delta SBP$ (equal angles)

ii) $\frac{SP}{BP} = \frac{AP}{SP}$ (corresponding angles)

$SP^2 = AP.BP$

Similarly, $TP^2 = AP.BP$

$\therefore ST = TP$

iii) In ΔDSP and ΔDTP,

$ST = TP$ (proven)

DP is common

$\angle SPD = \angle TPD$ (DP is bisector of $\angle P$)

$\therefore \Delta DSP \equiv \Delta DTP$ (SAS)

$\therefore \angle DSP = \angle DTP = 90°$ (corresponding angles)|

$\therefore TD$ passes through the centre (the radius is perpendicular to the tangent at the point of contact)

c)

i) $P_{n-1} = \cos\frac{\alpha}{2}\cos\frac{\alpha}{4}...\cos\frac{\alpha}{2^{n-1}}$

$P_n \sin\frac{\alpha}{2^n} = \cos\frac{\alpha}{2}\cos\frac{\alpha}{4}...\cos\frac{\alpha}{2^{n-1}}\cos\frac{\alpha}{2^n}\sin\frac{\alpha}{2^n}$

$= P_{n-1}\frac{1}{2}\sin 2\left(\frac{\alpha}{2^n}\right)$

$= \frac{1}{2}P_{n-1}\sin\frac{\alpha}{2^{n-1}}$

ii) $P_n \sin\frac{\alpha}{2^n} = \frac{1}{2}P_{n-1}\sin\frac{\alpha}{2^{n-1}}$

$= \frac{1}{2^2}P_{n-2}\sin\frac{\alpha}{2^{n-2}}$

$= \frac{1}{2^{n-1}}P_1\sin\frac{\alpha}{2}$

$= \frac{1}{2^{n-1}}\cos\frac{\alpha}{2}\sin\frac{\alpha}{2}$

$= \frac{1}{2^n}\left(2\cos\frac{\alpha}{2}\sin\frac{\alpha}{2}\right)$

$$= \frac{1}{2^n}\sin\alpha$$

$$\therefore P_n = \frac{\sin\alpha}{2^n \sin\frac{\alpha}{2^n}}$$

iii) $\sin\frac{\alpha}{2^n} < \frac{\alpha}{2^n}$

$2^n \sin\frac{\alpha}{2^n} < \alpha$

But $2^n \sin\frac{\alpha}{2^n} = \frac{\sin\alpha}{P_n}$

$$\therefore \frac{\sin\alpha}{\cos\frac{\alpha}{2}\cos\frac{\alpha}{2^2}...\cos\frac{\alpha}{2^n}} < \alpha$$

QUESTION 8

a)

i) $\omega^3 = 1, \therefore$ If k is a multiple of 3, $1+\omega^k+\omega^{2k} = 3$

If k is not a multiple of 3,

$\omega^3 - 1 = (\omega-1)(1+\omega+\omega^2) = 0$

$\left(\omega^k\right)^3 - 1 = \left(\omega^k - 1\right)\left(1+\omega^k+\omega^{2k}\right) = 0$

Since $\omega^k - 1 \neq 0, 1+\omega^k+\omega^{2k} = 0$

ii) $(1+\omega)^n = \binom{n}{0} + \binom{n}{1}\omega + \binom{n}{2}\omega^2 + ... + \binom{n}{n}\omega^n$

$(1+\omega^2)^n = \binom{n}{0} + \binom{n}{1}\omega^2 + \binom{n}{2}\omega^4 + ... + \binom{n}{n}\omega^{2n}$

iii) $(1+1)^n = \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + ... + \binom{n}{n}$

$\therefore (1+\omega)^n + (1+\omega^2)^n + (1+1)^n = 3\binom{n}{0} + \binom{n}{1}(1+\omega+\omega^2)$

$+\binom{n}{2}(1+\omega^2+\omega^4) + \binom{n}{3}(1+\omega^3+\omega^6) + ...$

$+\binom{n}{n}(1+\omega^n+\omega^{2n})$

$= 3\binom{n}{0} + 3\binom{n}{3} + ... + 3\binom{n}{3\ell}$, from (i)

$\therefore \binom{n}{0} + \binom{n}{3} + ... + \binom{n}{3\ell} = \frac{1}{3}\left(2^n + (1+\omega)^n + (1+\omega^2)^n\right)$

iv) If $n = 6M, M$ is an integer

$$\text{LHS} = \frac{1}{3}\left(2^n + (1+\omega)^{6M} + (1+\omega^2)^{6M}\right)$$

$$= \frac{1}{3}\left(2^n + (-\omega^2)^{6M} + (-\omega)^{6M}\right)$$

$$= \frac{1}{3}\left(2^n + (\omega^6)^{2M} + (\omega^6)^M\right)$$

$$= \frac{1}{3}\left(2^n + 1 + 1\right)$$

$$= \frac{1}{3}\left(2^n + 2\right)$$

b)

i) $I_n = \frac{q^{2n}}{n!}\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} \left(\frac{\pi^2}{4} - x^2\right)^n \cos x dx$

$u = \left(\frac{\pi^2}{4} - x^2\right)^n, du = n\left(\frac{\pi^2}{4} - x^2\right)^{n-1}(-2x)dx$

$dv = \cos x dx, v = \sin x$

$$I_n = \frac{q^{2n}}{n!}\left[\left(\frac{\pi^2}{4} - x^2\right)^n \sin x\right]_{\frac{-\pi}{2}}^{\frac{\pi}{2}} + \frac{2nq^{2n}}{n!}\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} \left(\frac{\pi^2}{4} - x^2\right)^{n-1} x\sin x dx$$

$$= \frac{2q^{2n}}{(n-1)!}\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} \left(\frac{\pi^2}{4} - x^2\right)^{n-1} x\sin x dx$$

$u = \left(\frac{\pi^2}{4} - x^2\right)^{n-1} x,$

$\frac{du}{dx} = (n-1)\left(\frac{\pi^2}{4} - x^2\right)^{n-2}(-2x^2) + \left(\frac{\pi^2}{4} - x^2\right)^{n-1}$

$dv = \sin x dx, v = -\cos x$

$$I_n = \frac{2q^{2n}}{(n-1)!}\left[\left(\frac{\pi^2}{4} - x^2\right)^{n-1}(-x\cos x)\right]_{\frac{-\pi}{2}}^{\frac{\pi}{2}} - \frac{4q^{2n}}{(n-2)!}\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} \left(\frac{\pi^2}{4} - x^2\right)^{n-2} x^2\cos x dx + \frac{2q^{2n}}{(n-1)!}\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} \left(\frac{\pi^2}{4} - x^2\right)^{n-1} \cos x dx$$

$$= \frac{2q^{2n}}{(n-1)!}\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}}\left(\frac{\pi^2}{4}-x^2\right)^{n-1}\cos x dx$$

$$-\frac{4q^{2n}}{(n-2)!}\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}}\left(\frac{\pi^2}{4}-x^2\right)^{n-2}x^2\cos x dx$$

ii) $I_n = 2q^2 I_{n-1}$

$$-\frac{4q^{2n}}{(n-2)!}\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}}\left(\frac{\pi^2}{4}-x^2\right)^{n-2}\left(\frac{\pi^2}{4}-\left(\frac{\pi^2}{4}-x^2\right)\right)\cos x dx$$

$$= 2q^2 I_{n-1} - 4q^4\frac{\pi^2}{4}I_{n-2} + 4q^2(n-1)I_{n-1}$$

$$= (4n-2)q^2 I_{n-1} - \pi^2 q^4 I_{n-2}$$

$$= (4n-2)q^2 I_{n-1} - \frac{p^2}{q^2}q^4 I_{n-2}$$

$$= (4n-2)q^2 I_{n-1} - p^2 q^2 I_{n-2}$$

iii) Since I_0 and I_1 are integers (given) and all "coefficients" of I_{n-1} and I_{n-2} are integers it follows that I_n is an integer.

iv) For $-\frac{\pi}{2} \le x \le \frac{\pi}{2}, \frac{\pi^2}{4} - x^2 \le \frac{\pi^2}{4}$ and $0 \le \cos x \le 1$, so, clearly $I_n > 0$

Also, $I_n < \frac{q^{2n}}{n!}\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}}\frac{\pi^{2n}}{4^n}dx = \frac{q^{2n}}{n!}\frac{\pi^{2n+1}}{4^n}$

$$= \frac{q^{2n}}{n!}\frac{p^{2n+1}}{q^{2n+1}4^n}$$

$$= \frac{p^{2n+1}}{q}\frac{1}{2^{2n}}\frac{1}{n!}$$

$$= \left(\frac{p}{q}\right)\left(\frac{p}{2}\right)^{2n}\frac{1}{n!}$$

v) In part (iv) we have proven that $\left(\frac{p}{q}\right)\left(\frac{p}{2}\right)^{2n}\frac{1}{n!}$ is a decimal between 0 and 1, in part (iii) we have proven that it is an integer. That's impossible, $\therefore \pi \neq \frac{p}{q}$.

*Note: The question gave $I_1 = 4q$ and told us not to prove it. In fact,

$$I_1 = \frac{q^2}{1!}\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}}\left(\frac{\pi^2}{4}-x^2\right)\cos x dx$$

$$= q^2 \times 4$$

$$= 4q^2$$