2015 NSW BOS Mathematics Extension 1 Solutions

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Section I

1	2	3	4	5	6	7	8	9	10
A	C	B	C	A	D	B	B	D	C

Q1 $\text{Remainder} = (-3)^3 - 6(-3) = -9$ A

Q2 $\dfrac{dN}{dt} = k80e^{kt} = k(N-100)$ C

Q3 $(3+x)x = (6+4)\times 4,\ x > 0$

$x^2 + 3x - 40 = 0,\ (x-5)(x+8) = 0,\ \therefore\ x = 5$ B

Q4 C

Q5 $y = \dfrac{3x}{(x+1)(x+2)} = \dfrac{\frac{3}{x}}{(1+\frac{1}{x})(1+\frac{2}{x})}$, as $x \to \pm\infty,\ y \to 0$ A

Q6 $-1 \le 2x \le 1,\ -\dfrac{1}{2} \le x \le \dfrac{1}{2}$ D

Q7 $\left[\sin^{-1}\left(\dfrac{x}{2}\right)\right]_0^k = \dfrac{\pi}{3},\ \sin^{-1}\left(\dfrac{k}{2}\right) = \dfrac{\pi}{3},\ \dfrac{k}{2} = \dfrac{\sqrt{3}}{2},\ k = \sqrt{3}$ B

Q8 $\lim\limits_{x\to 3} \dfrac{\frac{\sin(x-3)}{(x-3)}}{x+2} \to \dfrac{1}{5}$ B

Q9 $a = 2\times 3 = 6;\ \dfrac{2\pi}{n} = \dfrac{1}{2}\times\dfrac{2\pi}{4},\ \therefore\ n = 8$ D

Q10 $\cos\left(2t - \dfrac{\pi}{3}\right) = 0,\ 2t - \dfrac{\pi}{3} = \dfrac{\pi}{2}, \dfrac{3\pi}{2},\ 2t = \dfrac{5\pi}{6}, \dfrac{11\pi}{6}$

$\therefore\ t = \dfrac{5\pi}{12}, \dfrac{11\pi}{12},\ \therefore\ P\left(\dfrac{11\pi}{12}, 0\right)$ C

Section II

Q11a $\displaystyle\int \sin^2 x\,dx = \int \frac{1}{2}(1-\cos 2x)\,dx = \frac{1}{2}x - \frac{1}{4}\sin 2x + c$

Q11b $\tan\theta = \dfrac{(-3)-(2)}{1+(-3)(2)} = \dfrac{-5}{1-6} = 1,\ \theta = \dfrac{\pi}{4}\quad (45°)$

Q11c $\dfrac{4}{x+3} \ge 1,\ \dfrac{4}{x+3} - 1 \ge 0$ where $x+3 > 0,\ \therefore\ x > -3$

For $x+3 > 0,\ 4 \ge x+3,\ x \le 1\ \therefore\ -3 < x \le 1$

Q11d

$A\cos(x+\alpha) = A\cos x\cos\alpha - A\sin x\sin\alpha = 5\cos x - 12\sin x$

$\therefore\ A\cos\alpha = 5$ and $A\sin\alpha = 12\ \therefore\ A = 13$ and $\alpha = \tan^{-1}\left(\frac{12}{5}\right)$

$\therefore\ 5\cos x - 12\sin x = 13\cos\left(x + \tan^{-1}\left(\frac{12}{5}\right)\right)$

Q11e Let $u = 2x-1,\ x = \dfrac{u+1}{2},\ dx = \dfrac{1}{2}du$

When $x = 1,\ u = 1$; when $x = 2,\ u = 3$

$\displaystyle\int_1^2 \frac{x}{(2x-1)^2}\,dx = \int_1^3 \frac{u+1}{4u^2}\,du = \frac{1}{4}\int_1^3 \left(\frac{1}{u} + \frac{1}{u^2}\right)du$

$= \dfrac{1}{4}\left[\ln u - \dfrac{1}{u}\right]_1^3 = \dfrac{1}{4}\left(\ln 3 + \dfrac{2}{3}\right)$

Q11fi $P(x) = x^3 - kx^2 + 5x + 12$

$P(3) = 3^3 - k3^2 + 5\times 3 + 12 = 0,\ \therefore\ k = 6$

Q11fii $P(x) = x^3 - 6x^2 + 5x + 12$

$= (x-3)(x^2 - 3x - 4) = (x-3)(x-4)(x+1)$

Hence the zeros of $P(x)$ are: $^-1, 3$ and 4

Q12ai $\angle ACB = 90° - 30° = 60°$

Q12aii $\angle ADX = \angle ACD = 30°$

Q12aiii $\angle CAB = \angle CDB$

(angles on the circumference subtended by the same arc CB)

$\angle CDB = 100° - 30° = 70°$

(interior and exterior angles of a triangle)

$\therefore\ \angle CAB = 70°$

Q12bi $(p+q)x - 2y - 2apq = 0$ passes through the focus $(0, a)$

$\therefore\ (p+q)0 - 2a - 2apq = 0,\ \therefore\ pq = -1$

Q12bii $2ap = 8a,\ \therefore\ p = 4$ and $\therefore\ q = \dfrac{-1}{4}\ \therefore\ Q\left(\dfrac{-a}{2}, \dfrac{a}{16}\right)$

Q12ci $\Delta OAM:\ \dfrac{OA}{OM} = \cot 15°,\ OA = OM\cot 15° = h\cot 15°$

Q12cii $\Delta OBM:\ \dfrac{OB}{OM} = \cot 13°,\ OB = OM\cot 13° = h\cot 13°$

$\Delta OAB:\ OB^2 - OA^2 = AB^2,\ h^2(\cot^2 13° - \cot^2 15°) = 2000^2$

$h = \dfrac{2000}{\sqrt{\frac{1}{\tan^2 13°} - \frac{1}{\tan^2 15°}}} \approx 910$

Q12di The cosine rule: $160^2 = r^2 + r^2 - 2r^2\cos\theta$

$\therefore\ 160^2 = 2r^2(1-\cos\theta)$

Q12dii $r\theta = 200,\ r = \dfrac{200}{\theta}$

$160^2 = 2r^2(1-\cos\theta),\ 160^2 = 2\left(\dfrac{200}{\theta}\right)^2(1-\cos\theta)$

$160^2\theta^2 = 2(200)^2(1-\cos\theta),\ \therefore\ 8\theta^2 + 25\cos\theta - 25 = 0$

Q12diii Let $f(\theta) = 8\theta^2 + 25\cos\theta - 25,\ f'(\theta) = 16\theta - 25\sin\theta$

$\theta_1 = \pi,\ \theta_2 = \theta_1 - \dfrac{f(\theta_1)}{f'(\theta_1)} = \pi - \dfrac{8\pi^2 + 25\cos\pi - 25}{16\pi - 25\sin\pi} \approx 2.57$

Q13ai The particle is at rest ($v=0$) for $x=3$, 7.

Q13aii Maximum speed $=\sqrt{11}\text{ m s}^{-1}$

Q13aiii The amplitude of the oscillation is $a=2$.
Maximum speed occurs at the centre of the oscillation, $\therefore\ c=5$

$\therefore\ v^2=n^2\left(4-(x-5)^2\right)\ \therefore\ 11=4n^2,\ n=\dfrac{\sqrt{11}}{2}$

Q13bi The third term is $\binom{18}{2}(2x)^{16}\left(\dfrac{1}{3x}\right)^2=\binom{18}{2}\dfrac{2^{16}}{3^2}x^{14}$

$\therefore\ a_2=\binom{18}{2}\dfrac{2^{16}}{3^2}$

Q13bii The term independent of x is
$\binom{18}{9}(2x)^9\left(\dfrac{1}{3x}\right)^9=\binom{18}{9}\dfrac{2^9}{3^9}=\binom{18}{9}\left(\dfrac{2}{3}\right)^9$

Q13c For $n=1$, $\dfrac{1}{2!}=1-\dfrac{1}{2!}$ is true

Assume it is true for $n=k$,
i.e. $\dfrac{1}{2!}+\dfrac{2}{3!}+\dfrac{3}{4!}+\cdots+\dfrac{k}{(k+1)!}=1-\dfrac{1}{(k+1)!}$

For $n=k+1$,

$\dfrac{1}{2!}+\dfrac{2}{3!}+\dfrac{3}{4!}+\cdots+\dfrac{k}{(k+1)!}+\dfrac{k+1}{(k+2)!}=1-\dfrac{1}{(k+1)!}+\dfrac{k+1}{(k+2)!}$

$=1-\dfrac{1}{(k+1)!}+\dfrac{k+1}{(k+2)!}=1-\dfrac{k+2}{(k+2)!}+\dfrac{k+1}{(k+2)!}=1-\dfrac{1}{(k+2)!}$

$\therefore\ \dfrac{1}{2!}+\dfrac{2}{3!}+\dfrac{3}{4!}+\cdots+\dfrac{n}{(n+1)!}=1-\dfrac{1}{(n+1)!}$ for all integers $n\geq 1$.

Q13di $f(x)=\cos^{-1}(x)+\cos^{-1}(-x)$, $-1\leq x\leq 1$

$f'(x)=\dfrac{-1}{\sqrt{1-x^2}}+\dfrac{(-1)(-1)}{\sqrt{1-x^2}}=0$, $\therefore\ f(x)$ is constant

Q13dii Let $x=1$, $\cos^{-1}(1)+\cos^{-1}(-1)=0+\pi=\pi$

$\therefore\ f(x)=\cos^{-1}(x)+\cos^{-1}(-x)=\pi$, $\therefore\ \cos^{-1}(-x)=\pi-\cos^{-1}(x)$

Q14ai Given $x=Vt\cos\theta$, $y=Vt\sin\theta-\dfrac{1}{2}gt^2$

Let $y=0$, $Vt\sin\theta-\dfrac{1}{2}gt^2=t\left(V\sin\theta-\dfrac{1}{2}gt\right)=0$

$\therefore\ t=0$ or $t=\dfrac{2V\sin\theta}{g}$

$\therefore\ x=0$ or $x=\dfrac{V^2 2\sin\theta\cos\theta}{g}=\dfrac{V^2\sin 2\theta}{g}$ respectively

$\therefore$ the range is $\dfrac{V^2\sin 2\theta}{g}$

Q14aii When $t=\dfrac{2V}{\sqrt{3}g}$ and $\theta=\dfrac{\pi}{3}$

$\dfrac{dx}{dt}=V\cos\dfrac{\pi}{3}=\dfrac{V}{2}$ and $\dfrac{dy}{dt}=V\sin\dfrac{\pi}{3}-\dfrac{2V}{\sqrt{3}}=\dfrac{\sqrt{3}V}{2}-\dfrac{2V}{\sqrt{3}}=\dfrac{-V}{2\sqrt{3}}$

$\therefore\ \tan\phi=\dfrac{dy}{dx}=\dfrac{\frac{dy}{dt}}{\frac{dx}{dt}}=-\dfrac{1}{\sqrt{3}}$, $\therefore\ \phi=-\dfrac{\pi}{6}$

The projectile travels at angle $\dfrac{\pi}{6}$ with (below) the horizontal.

Q14aiii The negative sign in $\phi=-\dfrac{\pi}{6}$ indicates the particle is travelling downwards.

Q14bi $\ddot{x}=x-1$, $\dfrac{d}{dx}\left(\dfrac{\dot{x}^2}{2}\right)=x-1$,

$\dfrac{\dot{x}^2}{2}=\int(x-1)dx=\dfrac{(x-1)^2}{2}+c_1$

To satisfy the initial conditions, $\dot{x}=1$ when $x=0$, $c_1=0$ and
$\dot{x}^2=(x-1)^2$, $\therefore\ \dot{x}=-(x-1)$, i.e. $\dot{x}=1-x$

Q14bii $\dot{x}=1-x$, $\dfrac{dx}{dt}=1-x$, $\dfrac{dt}{dx}=\dfrac{1}{1-x}$, $t=\int\dfrac{1}{1-x}dx$,

$t=-\ln|1-x|+c_2$

$x=0$ when $t=0$, $\therefore\ c_2=0$ and $t=-\ln|1-x|$, $\therefore\ 1-x=e^{-t}$

$\therefore\ x=1-e^{-t}$

Q14biii $\lim\limits_{t\to\infty}x=1$

Q14ci $\Pr(A\text{ wins the seventh game and any 4 of the first 6 games})$

$=\binom{6}{4}\left(\dfrac{1}{2}\right)^7$

Q14cii $\binom{4}{4}\left(\dfrac{1}{2}\right)^5+\binom{5}{4}\left(\dfrac{1}{2}\right)^6+\binom{6}{4}\left(\dfrac{1}{2}\right)^7$

Q14ciii Player A must win a total of $(n+1)$ games in at most $(2n+1)$ games.
If A did not win the $(2n+1)$th game, B gets the prize.

$\Pr(A\text{ gets the prize})=\dfrac{1}{2}$

$\binom{n}{n}\left(\dfrac{1}{2}\right)^{n+1}+\binom{n+1}{n}\left(\dfrac{1}{2}\right)^{n+2}+\binom{n+2}{n}\left(\dfrac{1}{2}\right)^{n+3}+\cdots+\binom{2n}{n}\left(\dfrac{1}{2}\right)^{2n+1}=\dfrac{1}{2}$

Multiply both sides by 2^{2n+1} :

$\binom{n}{n}2^n+\binom{n+1}{n}2^{n-1}+\binom{n+2}{n}2^{n-2}+\cdots+\binom{2n}{n}=2^{2n}$

Please inform mathline@itute.com re conceptual and/or mathematical errors.

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