2014 HSC Examination

Mathematics Extension 1

SOLUTIONS

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Multiple Choice

1. D
2. A
3. C
4. D
5. B
6. B
7. A
8. D
9. C
10. C

Brief Explanations

Question 1 Angle at the centre is twice the angle at the circumference on the same arc.

Question 2 Standard Auxiliary Angle Formula problem.

Question 3 Obtain the general term, let the power of x be zero, then solve & substitute.

Question 4 Standard angle between two lines problem.

Question 5 Use the product of the roots, and the sum in pairs to eliminate options.

Question 6 Standard differentiation problem.

Question 7 Amplitude of 5 leaves (A) or (C). Only (A) satisfies the period.

Question 8 Choose them first with $\binom{15}{6}$ then permute with $5!$. Expand and simplify.

Question 9 Substitute $x=-1$ and it becomes a standard remainder theorem problem.

Question 10 Standard locus problem.

Written Response

Question 11 (a)

Let $u = x + \frac{2}{x}$, so our equation is $u^2 - 6u + 9 = 0$.

$$(u-3)^2 = 0$$

$$u = 3$$

$$x + \frac{2}{x} = 3$$

$$x^2 + 2 = 3x$$

$$x^2 - 3x + 2 = 0$$

$$(x-2)(x-1) = 0$$

$$x = 1 \text{ or } 2$$

Question 11 (b)

$P(\text{rain}) = 0.1$

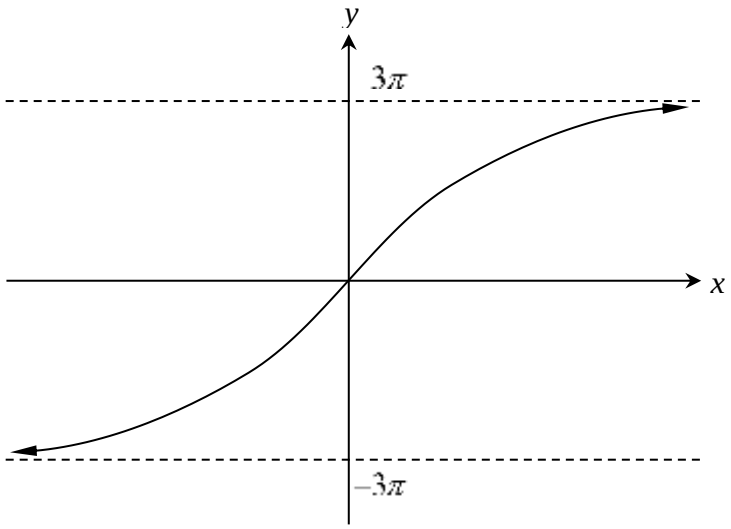
$P(\text{no rain}) = 0.9$

$$P(\text{fewer than 3 days of rain}) = P(\text{no rain}) + P(\text{1 day of rain}) + P(\text{2 days of rain})$$

$$= 0.9^{30} + \binom{30}{1} \times 0.1 \times 0.9^{29} + \binom{30}{2} \times 0.1^2 \times 0.9^{28}$$

Question 11 (c)

The curve $y = 6\tan^{-1} x$ has two horizontal asymptotes $y = 6 \times \left(\pm \frac{\pi}{2} \right) = \pm 3\pi$. Other than that, it looks like the standard inverse tan function.



Question 11 (d)

Note that the inverse of the substitution $x = u^2 + 1$ is not a function. Also note that we have the restriction $x > 1$. We will arbitrarily choose the restriction $u > 0$ in order to make our choice of limits. In actuality, it does not matter whether we choose $u > 0$ or $u < 0$ because the value of the indefinite integral is the same either way, due to the even nature of our integrand.

$x = u^2 + 1 \Rightarrow dx = 2u\,du$.

So we have our limits

$x = 5 \Rightarrow u = 2$

$x = 2 \Rightarrow u = 1$

And then substitute

$$I = \int_1^2 \frac{u^2 + 1}{u} \times 2u\,du$$

$$= 2\int_1^2 u^2 + 1\,du$$

$$= 2\left[\frac{u^3}{3} + u \right]_1^2$$

$$= 2\left[\frac{14}{3} - \frac{4}{3} \right]$$

$$= \frac{20}{3}$$

Question 11 (e)

Multiply both sides by x^2 first.

$$x\left(x^2+5\right)>6x^2$$

$$6x^2-x\left(x^2+5\right)<0$$

$$x\left(6x-x^2-5\right)<0$$

$$x\left(x^2-6x+5\right)>0$$

$$x\left(x-1\right)\left(x-5\right)>0$$

Upon sketching the cubic polynomial and observing where it is above the *x* axis, we get our solution set $0<x<1$ or $x>5$.

Question 11 (f)

We first compute the derivative of the numerator using the product rule.

$$\frac{d}{dx}\left(e^x\ln x\right)=e^x\ln x+\frac{e^x}{x}$$

We can now proceed to use the quotient rule to differentiate our expression.

$$\frac{d}{dx}\left(\frac{e^x\ln x}{x}\right)=\frac{x\left(e^x\ln x+\dfrac{e^x}{x}\right)-e^x\ln x}{x^2}$$

$$=\frac{xe^x\ln x+e^x-e^x\ln x}{x^2}$$

$$=\frac{e^x}{x^2}\left(x\ln x-\ln x+1\right)$$

Question 12 (a) (i)

Observe that the particle begins at the origin so when it first returns to the origin, it has travelled two laps of the amplitude, which is a distance of 4 metres.

Question 12 (a) (ii)

Let the particle satisfy the differential equation $\ddot{x} = -n^2 x$, as it is centred at the origin. From the displacement time equation, the value of *n* is 3, so the particle satisfies $\ddot{x} = -9x$.

The particle is first at rest when it is at the positive amplitude, which is when $x = 2$.

Hence, the acceleration is $\ddot{x} = -9(2) = -18 \text{ms}^{-2}$.

Question 12 (b)

$$V = \pi \int_0^{\frac{\pi}{8}} \cos^2 4x \, dx$$

$$= \frac{\pi}{2} \int_0^{\frac{\pi}{8}} (1 + \cos 8x) \, dx$$

$$= \frac{\pi}{2} \left[x + \frac{1}{8} \sin 8x \right]_0^{\frac{\pi}{8}}$$

$$= \frac{\pi}{2} \left[\frac{\pi}{8} + \frac{1}{8} \sin \pi \right]$$

$$= \frac{\pi^2}{16}$$

Question 12 (c)

We are given $\frac{d}{dx}\left(\frac{1}{2}v^2\right) = 2 - e^{-\frac{x}{2}}$.

Integrate both sides with respect to *x*.

$$\frac{1}{2}v^2 = 2x + 2e^{-\frac{x}{2}} + C$$

Substitute in $x = 0$, $v = 4$ to get $\frac{16}{2} = 2 + C \Rightarrow C = 6$.

$$\frac{1}{2}v^2 = 2x + 2e^{-\frac{x}{2}} + 6$$

$$v^2 = 4x + 4e^{-\frac{x}{2}} + 12$$

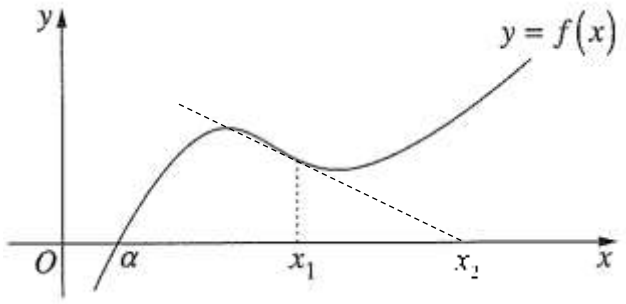
Question 12 (d)

Consider the binomial expansion $(1+x)^n = \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \binom{n}{3}x^3 + ... + \binom{n}{n}x^n$.

Substitute in $x = -1$ to obtain the result immediately.

Question 12 (e)

Note that from the mechanics behind Newton's Method, the x intercept of the tangent at $x = x_1$ will be our next approximation $x = x_2$.



When we draw a tangent at $x = x_1$, it will intersect the x axis at $x = x_2$, which is clearly further away from $x = \alpha$ than $x = x_1$.

Question 12 (f)

Notice that as $t \to \infty$, we have $T \to A$ since the other term tends to zero. However, the temperature T of the milk has to approach the temperature of the room, so $A = 23$.

We are given that the initial temperature of the milk is 2° C so when $t = 0$, $T = 2$.

$$T = 23 - Be^{-0.03t}$$

$$2 = 23 - B$$

$$B = 21$$

Hence the equation is $T = 23 - 21e^{-0.03t}$. We now let $T = 10$ and then solve for time.

$$10 = 23 - 21e^{-0.03t}$$

$$e^{-0.03t} = \frac{13}{21}$$

$$-0.03t = \ln\left(\frac{13}{21}\right)$$

$$t = \frac{\ln\left(\frac{21}{13}\right)}{0.03}$$

$$\approx 16 \text{ minutes}$$

Question 13 (a)

Base Case: $n=1$

$2^1+(-1)^2=2+1=3$, which is of course divisible by 3.

Inductive Hypothesis: $n=k$

$2^k+(-1)^{k+1}=3M$, where M is some integer.

ie: $(-1)^{k+1}=3M-2^k$

Inductive Step: $k \Rightarrow k+1$

$$2^{k+1}+(-1)^{k+2}=2^{k+1}+(-1)^k$$

$$=2^{k+1}-(-1)^{k+1}$$

$$=2^{k+1}-(3M-2^k)$$

$$=2^{k+1}+2^k-3M$$

$$=2^k(2+1)-3M$$

$$=3\left[2^k-M\right] \quad ...(\text{which is a number divisible by } 3)$$

Question 13 (b) (i)

From the diagram, we can see that $L^2=x^2+40^2$

By differentiating both sides with respect to x, and using the Chain Rule, we have

$$2L\frac{dL}{dx}=2x$$

$$\frac{dL}{dx}=\frac{x}{L}$$

$$=\cos\theta$$

Question 13 (b) (ii)

$$\frac{dL}{dt} = \frac{dL}{dx} \times \frac{dx}{dt}$$

$$= \cos\theta \times 3 \qquad \ldots\left(\text{since the truck moves to the right at a constant } 3\text{ms}^{-1}\right)$$

$$= 3\cos\theta$$

Question 13 (c) (i)

We have $P\left(2at, at^2\right)$ and $S\left(0, a\right)$, where the ratio that Q divides PS is $t^2 : 1$.

So we have $x_Q = \dfrac{(1)2at + \left(t^2\right)(0)}{1+t^2} = \dfrac{2at}{1+t^2}$.

Similarly, $y_Q = \dfrac{(1)at^2 + \left(t^2\right)a}{1+t^2} = \dfrac{2at^2}{1+t^2}$.

Question 13 (c) (ii)

We simply use Gradient = $\dfrac{\text{Rise}}{\text{Run}}$.

$$m_{OQ} = \frac{y}{x} = \frac{\dfrac{2at^2}{1+t^2}}{\dfrac{2at}{1+t^2}} = t.$$

Question 13 (c) (iii)

Substitute this into say the x coordinate of Q.

$$x = \frac{2a\left(\dfrac{y}{x}\right)}{1+\left(\dfrac{y}{x}\right)^2}$$

$$= \frac{2axy}{x^2+y^2}$$

$$x^2 + y^2 = 2ay \quad ...(\text{if } x \neq 0)$$

$$x^2 + y^2 - 2ay = 0$$

$$x^2 + (y-a)^2 = a^2$$

This is the equation of a circle centred at the focus with radius a, where $x \neq 0$. So the 'missing points' at the moment are $(0,0)$ and $(0,2a)$.

Now, we consider the case when $x = 0$. When this occurs, we have Q lying on the origin, which is a point satisfying the locus equation.

Hence, the equation of the locus is the circle $x^2 + (y-a)^2 = a^2$, except $(0,2a)$.

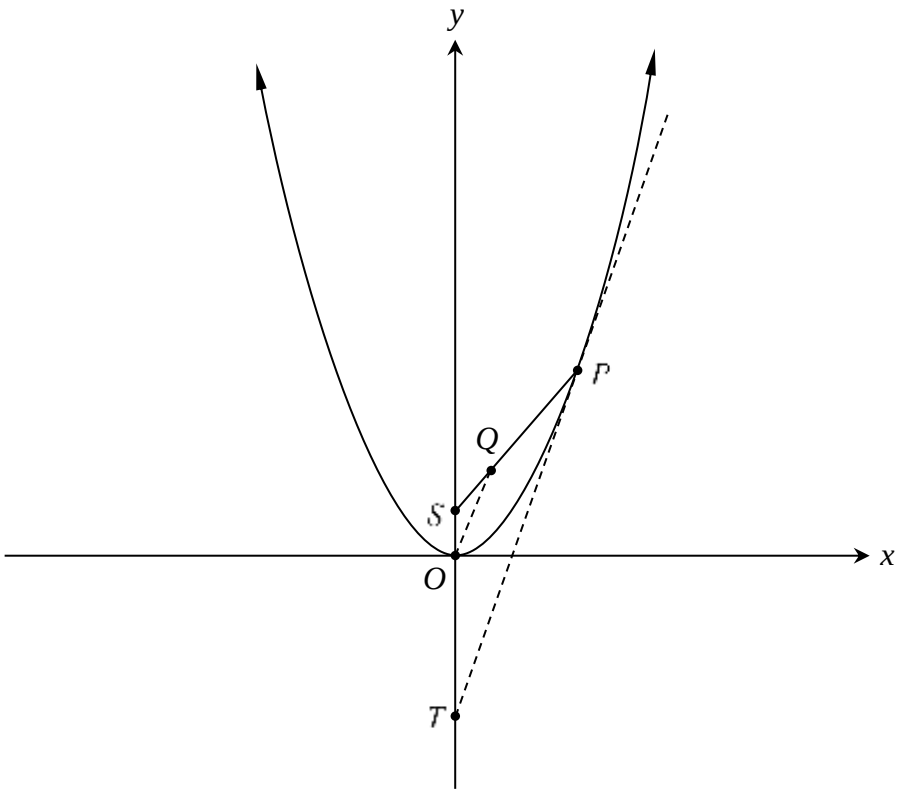
Geometric Argument

An alternative geometric argument can be made using the Reflection Property and (ii). For the sake of clarity as to not get lost within the working out, we will assume some results without proof.

Since $m_{OQ} = t$, we can say that it is parallel to the tangent at *P*, which also has gradient *t*.

Draw and extend the tangent at *P* such that it intersects the *y* axis at *T*.

One of the results used in the proof of the Reflection Property of the parabola is that ΔPST is isosceles with $\angle SPT = \angle STP$ (easily proven using the focus-directrix property of the parabola), which means that ΔSOQ is isosceles with $QS = OS = a$.



So we have $QS = a$, where *S* is a fixed point. However, *Q* is a variable point, meaning that *Q* moves in a manner such that it is always *a* units from the focus *S*.

In other words, *Q* is the circle centred at the focus *S* with radius *a*, which was the required result.

Note that any other proof of *QS* (ie: distance formula or a result analogous to $PS = PM$) being equal to *a* is also valid to demonstrate the result.

Question 13 (d) (i)

$\angle CQP$ is the opposite exterior angle of the cyclic quadrilateral $ABQP$ to the interior angle , $\angle BAC$, hence they are equal.

Question 13 (d) (ii)

First, construct *OP* to a point *D*, external to the circle.

Since *OP* and *OA* are the radii of circle $ABQP$, we can say that $\angle OAP = \angle OPA$ (angles opposite equal sides).

But $\angle OPA = \angle DPC$ (vertically opposite angles).

But since $\angle OAP = \angle CQP$ (from part (i)), we have $\angle CQP = \angle DPC$.

Hence, by the converse of the Alternate Segment Theorem, we have *OP* being tangential to the circle circumscribing points *P*, *Q* and *C*.

Question 14 (a) (i)

We eliminate *t* to obtain the Cartesian equation.

$$t = \frac{x}{V\cos\theta}$$

Substitute into the equation for *y*.

$$y = -\frac{1}{2}g\left(\frac{x}{V\cos\theta}\right)^2 + V\left(\frac{x}{V\cos\theta}\right)\sin\theta$$

$$= -\frac{1}{2V^2}gx^2\sec^2\theta + x\tan\theta$$

Question 14 (a) (ii)

We first obtain the coordinate of *P*. However, notice that the equation of the ramp is $y = x\tan\left(-\frac{\pi}{4}\right) = -x$, since it is inclined at an angle of $-\frac{\pi}{4}$ from the horizontal.

So we solve the Cartesian equation simultaneously with the equation of the ramp.

$$-\frac{1}{2V^2}gx_P^{\,2}\sec^2\theta + x\tan\theta = -x_P$$

$$-\frac{1}{2V^2}gx_P\sec^2\theta + \tan\theta = -1 \qquad \text{...(since we are looking for } P \text{ ie: } x_P \neq 0)$$

$$\frac{1}{2V^2}gx_P\sec^2\theta = \tan\theta + 1$$

$$x_P = \frac{2V^2(\tan\theta+1)}{g\sec^2\theta}$$

$$= \frac{2V^2}{g}\cos^2\theta(\tan\theta+1)$$

$$= \frac{2V^2}{g}\cos\theta(\sin\theta+\cos\theta)$$

The distance *D* is the length of the 'hypotenuse', which is longer than the side lengths by a factor of $\sqrt{2}$.

Hence, we have $D = \sqrt{2}\times x_P = \frac{2\sqrt{2}V^2}{g}\cos\theta(\cos\theta+\sin\theta)$.

Question 14 (a) (iii)

First, we rearrange *D*.

$$D = \frac{2\sqrt{2}V^2}{g}\left(\cos^2\theta + \sin\theta\cos\theta\right)$$

$$= \frac{2\sqrt{2}V^2}{g}\left(\cos^2\theta + \frac{1}{2}\sin 2\theta\right)$$

Now we differentiate with respect to θ.

$$\frac{dD}{d\theta} = \frac{2\sqrt{2}V^2}{g}\left(-2\cos\theta\sin\theta + \cos 2\theta\right)$$

$$= \frac{2\sqrt{2}V^2}{g}\left(\cos 2\theta - \sin 2\theta\right)$$

Question 14 (a) (iv)

We use Auxiliary Angle Formulae to obtain this maximum.

$$\frac{dD}{d\theta} = \frac{2\sqrt{2}V^2}{g}\left(\cos 2\theta - \sin 2\theta\right)$$

$$= \frac{4V^2}{g}\cos\left(2\theta + \frac{\pi}{4}\right)$$

$$= 0 \qquad ...\left(\text{for the maximum/minimum value of } D\right)$$

Solving this, we get $2\theta + \frac{\pi}{4} = \frac{\pi}{2}$, yielding $\theta = \frac{\pi}{8}$. Notice that we reject the case when $2\theta + \frac{\pi}{4} = \frac{3\pi}{2}$ since it yields a solution that does not satisfy $0 \le \theta \le \frac{\pi}{2}$.

We now find the second derivative of D with respect to θ to verify that it is a maximum.

$$\frac{d^2D}{d\theta^2} = -\frac{8V^2}{g}\sin\left(2\theta + \frac{\pi}{4}\right)$$

$$= -\frac{8V^2}{g}\sin\left(\frac{\pi}{2}\right) \qquad ...\left(\text{when } \theta = \frac{\pi}{8}\right)$$

$$= -\frac{8V^2}{g}$$

$$< 0$$

Hence, D is maximised when $\theta = \frac{\pi}{8}$.

Question 14 (b) (i)

We compute the probabilities of the two cases and then add them.

Case #1: Player *A* wins on the first turn

This is the easy case when Player *A* spins the arrow, and it lands directly on *P*.

Hence, the probability is *p*.

Case #2: Player *A* wins on the second turn

Player *A* must miss their first turn by landing on *R*.

Then, Player *B* spins and the arrow lands on *Q*, thus making them lose and hence Player *A* winning on the second turn.

Hence, the probability is $r \times q = rq$.

Now that we've figured out all the cases, we can add their probabilities. However, first note that since the only possible outcomes are *P*, *Q* or *R*, we have $p + q + r = 1$, or in other words $q = 1 - p - r$.

Total probability is thus

$$\begin{aligned} p + rq &= p + r(1 - p - r) \\ &= p + r - rp - r^2 \\ &= (1 - r)(p + r) \end{aligned}$$

Question 14 (b) (ii)

Note that there are infinitely many ways for Player *A* to win the game. The probability of *A* 'eventually winning' is the total probability of all possible ways that Player *A* can win.

If we continue adding the cases of Player *A* winning due to Player *B* losing, then *A* wins on the second turn, then the fourth turn, then the sixth term etc…, and we obtain the probability

$$rq + r^3q + r^5q + ... \qquad (2)$$

However, observe that Player *A* can also win the game by actually landing on *P*, rather than winning because Player *B* is losing. So similarly to (i), Player *A* can win by…

Case #1: Landing at *P* on his first go

The probability is just *p*.

Case #2: Landing at *P* on his second go

Player *A* must miss their first turn, then so must *B*. Now that it is Player *A*'s turn again, they must land on *P*, so the probability is $r \times r \times p = r^2 p$.

Case #3: Landing at *P* on his third go

Player *A* must miss their first turn and so must *B*. This occurs twice, so we have $r \times r \times r \times r$.

Finally, Player *A* lands on *P*, so the probability is $r^4 p$

…

So continuing this pattern and adding their probabilities, we have

$$p + r^2 p + r^4 p + ... \qquad (1)$$

Hence, the probability of Player *A* eventually winning the game, in other words the total probability of Player *A* winning the game, is acquired by adding (1) and (2).

Note that the use of the limiting sum formula is justified since *r* is a probability ie: $|r^2| < 1$.

$$\begin{aligned}\left(rq + r^3q + r^5q + r^7q...\right) + \left(p + r^2p + r^4p + r^6p + ...\right) &= rq\left(1 + r^2 + r^4 + r^6 + ...\right) + p\left(1 + r^2 + r^4 + r^6 + ...\right) \\ &= \left(p + rq\right) \times \left(1 + r^2 + r^4 + r^6 + ...\right) \\ &= \left(1 - r\right)\left(p + r\right) \times \frac{1}{1 - r^2} \qquad ...\left(\text{using (i)}\right) \\ &= \frac{p + r}{1 + r}\end{aligned}$$

Alternatively

Let X be the sum of the probabilities of every single way that Player A can win (in other words, the probability of Player A 'eventually' winning).

Then the total probability of Player B winning, provided they have the first turn, should also be X by symmetry of the scenario.

Now observe that every single possible scenario can be categorised into three cases, noting that Player A has the first turn.

Case #1: Player A wins

The probability is just X, by definition.

Case #2: Player B wins eventually after Player A misses their first turn

The probability is rX since Player A must miss their first turn, then the scenario is now equivalent to Player B having the first turn.

Case #3: Player B wins because Player A lost on their first turn

The probability is q, again by definition.

So we have $X + rX + q = 1$, as we have now acquired a complete sample space.

Rearranging yields

$$X = \frac{1-q}{1+r}$$

$$= \frac{p+r}{1+r} \qquad \ldots(\text{since that } p+q+r=1)$$

As required.