HSC 2008 MATHEMATICS EXTENSION 1 (3 unit) EXAM : ANSWERS/SOLUTIONS.

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Question1

Q1.a

The usual polynomial division gives

$$\begin{array}{r} x^2-3x+9 \\ x+3\overline{)\;x^3} \\ \underline{-x^3-3x^2} \\ -3x^2 \\ \underline{3x^2+9x} \\ 9x \\ \underline{-9x-27} \\ -27 \end{array}$$

and so the remainder is -27.

Or using the horner scheme we have

$$\begin{array}{r|cccc} & 1 & 0 & 0 & 0 \\ -3 & & -3 & 9 & -27 \\ \hline & 1 & -3 & 9 & -27 \end{array}$$

which gives the same result.

Q1.b

$$\frac{d}{dx}\cos^{-1}3x = \frac{-1}{\sqrt{1-(3x)^2}} \times 3 = \frac{-3}{\sqrt{1-9x^2}}$$

Q1.c

$$\int_{-1}^{1} \frac{1}{\sqrt{4-x^2}}\,dx = \left[\sin^{-1}\frac{x}{2}\right]_{-1}^{1} = \sin^{-1}\frac{1}{2} - \sin^{-1}\frac{-1}{2} = \pi/6 - \frac{-\pi}{6} = \frac{\pi}{3}$$

Q1.d

$$(2x+3y)^12 = \sum_{r=0}^{12}\binom{12}{r}(2x)^r(3y)^{12-r} = \sum_{r=0}^{12}\binom{12}{r}2^r3^{12-r}x^ry^{12-r}$$

Thus we require the coefficient when $r=8$ which is

$$\binom{12}{8}2^83^{12-8} = \binom{12}{8}2^83^4 = 10264320$$

Q1.e

$$\int_0^{\pi/4}\cos\theta\sin^2\theta\,d\theta = \left[\frac{\sin^3\theta}{3}\right]_0^{\pi/4} = \frac{1}{3}\left(\frac{1}{\sqrt{2}}\right)^3 - 0 = \frac{1}{6\sqrt{2}} = \frac{\sqrt{2}}{12}$$

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Q1.f

The domain of the function $y = \log_e x$ is $x > 0$.
$\therefore$ the function $f(x) = \log_e[(x-3)(5-x)]$ has domain where x satisfies, $(x-3)(5-x) > 0$.
Solving this yields, $3 < x < 5$.

Question2

Q2.a

$$I = \int_e^{e^2} \frac{1}{x(\log_e x)^2} dx$$

Letting $u = \log_e x = \ln x$, $du = \frac{1}{x}dx$. Also then, when $x = e^2$, $u = \ln e^2 = 2$ and when $x = e$, $u = \ln e = 1$. So, $I = \int_1^2 \frac{du}{u^2} = \int_1^2 u^{-2} du = [-u]_1^2 = -1/2 + 1 = 0.5$

Q2.b

We are given that $v = 0$ at $x = 0$ when $t = 0$.

$$\begin{aligned}
\frac{d}{dx}\left(\frac{v^2}{2}\right) &= x + 4 \\
\frac{v^2}{2} &= x^2/2 + 4x + c \\
0 &= 1/2 + 4 + c \implies c = -4.5 \\
\frac{v^2}{2} &= x^2/2 + 4x - 4.5 \\
v^2 &= x^2 + 8x - 9
\end{aligned}$$

so when $x = 2$, speed $\therefore |v| = \sqrt{x^2 + 8x - 9} = \sqrt{2^2 + 8 \times 2 - 9} = \sqrt{11}$.

Q2.c

$p(x) = ax^3 + 16x^2 + cx - 120$, where $a, c \in \mathbb{C}$.
Sum Roots: $-2 + 3 + \alpha = -16/a \implies 1 + \alpha = -16/a$..(1)
Product of pairs of roots: $-2 \times 3 + -2 \times \alpha + 3 \times \alpha = c/a \implies \alpha - 6 = c/a$...(2)
Product of the three roots: $-2 \times 3 \times \alpha = 120/a \implies \alpha a = -20$...(3)
Hence using (1), (2), (3) we have,
$a + \alpha a = -16 \implies a - 20 = -16 \implies a = 4 \implies \alpha = -20/4 = -5$.

Q2.d

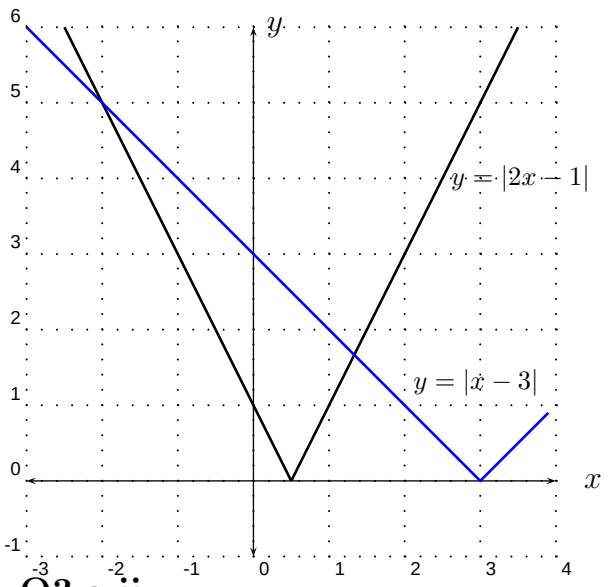
Newtons rule is $x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$.
We have $f(x) = \tan x - \ln x$, $f'(x) = \sec^2 x - 1/x$. So,
$x_1 = 4 - \frac{f(4)}{f'(4)} = 4 - \frac{\tan 4 - \ln 4}{\sec^2 4 - 0.25} = 4.11$.

Question3

Q3.a

Q3.a.i

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Q3.a.ii

Find the intersection points of the two graphs first, $|2x-1| = |x-3|$

$4x^2 - 4x + 1 = x^2 - 6x + 9$

$3x^2 + 2x - 8 = 0$

$(3x-4)(x+2) = 0 \implies x = 4/3, -2.$

By inspection of the graph we can now see that $|2x-1| \leq |x-3|$ when $-2 \leq x \leq \frac{4}{3}$.

Q3.b

$$S_n : 1 \times 3 + 2 \times 4 + 3 \times 5 + \cdots + n(n+2) = \frac{n(n+1)(2n+7)}{6}$$

Step 1: For $n = 1$, lhs$= 1 \times 3 = 3$.

rhs$= \dfrac{1(1+1)(2 \times 1 + 7)}{6} = 18/6 = 3 = lhs$. So the statement is true for $n = 1$.

Step 2: Assume that the statement is true for $n = k$, so we assume

$$S_k : 1 \times 3 + 2 \times 4 + 3 \times 5 + \cdots + k(k+2) = \frac{k(k+1)(2k+7)}{6} \qquad (1)$$

Step 3: We wish to prove that the statement is true for $n = k+1$ assuming (1) is true.

RTP:

$$S_{k+1} : 1 \times 3 + 2 \times 4 + 3 \times 5 + k(k+2) + (k+1)(k+3) = \frac{(k+1)(k+2)(2k+9)}{6} \qquad (2)$$

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$$\begin{aligned}
\text{LHS} &= \underbrace{1\times 3+2\times 4+3\times 5+k(k+2)}+(k+1)(k+3)\\
&= \frac{k(k+1)(2k+7)}{6}+(k+1)(k+3)\\
&= \frac{(k+1)}{6}[2k^2+7k+6k+18]\\
&= \frac{(k+1)}{6}[2k^2+13k+18]\\
&= \frac{(k+1)}{6}(k+2)(2k+9)\\
\\
&= \text{ RHS of statement } S_{n+1}
\end{aligned}$$

So the statement is true for $n = k+1$.

Step 4: We have shown the statement is true for $n = 1$ and its also true for $n = k+1$ whenever it is true for $n = k$ and hence it is true for $n = 1, 2, 3....$ That is it is true for all integers $n \geq 1$.

Q3.c

Q3.c.i $\tan\theta = x/l$ and so $\theta = \tan^{-1}(x/l)$

Differentiating,

$$\begin{aligned}
\frac{d\theta}{dt} &= \frac{1}{1+\dfrac{x^2}{l^2}}\times\frac{1}{l}.\frac{dx}{dt}\\
&= \frac{l^2}{l^2+x^2}\times\frac{1}{l}\times v\\
&= \frac{vl}{l^2+x^2}
\end{aligned}$$

Q3.c.ii

Clearly the maximum value of $\dfrac{vl}{l^2+x^2}$ occurs when $x = 0$, when the denominator is least. This can be verified using calculus and stationary points etc.

The maximum value is thus $vl/(l^2+0) = v/l$.

Q3.c.iii

Solve for x in terms of l;

$$\begin{aligned}
&\frac{vl}{(l^2+x^2)} = \frac{v}{4l}\\
&l^2+x^2 = 4l^2\\
&x^2 = 3l^2\\
&x = \pm\sqrt{3}\,l\\
&\therefore \tan\theta = \sqrt{3}l/l = \pm\sqrt{3}\\
&\therefore \theta = \pi/3, -\pi/3.
\end{aligned}$$

Question4

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Q4.a

Q4.a.i Initial condition:

When $t = 0$, $T(0) = 190 - 185e^0 = 190 - 185 = 5$.

The differential equation:

$$\begin{aligned} LHS = \frac{dT}{dt} &= (-185) \times (-k)e^{-kt} \\ &= 185ke^{-kt} \\ &= k(185e^{-kt}) \\ \frac{dT}{dt} &= k(190 - T) = RHS \text{ as required.} \end{aligned}$$

Q4.a.ii

When $t = 1$, $T = 29$ so $29 = 190 - 185e^{-k}$

$$e^{-k} = \frac{190 - 29}{185} \implies -k = \ln\left(\frac{161}{185}\right)$$

$$\implies k = \ln\left(\frac{185}{161}\right)$$

So updating our equation and substituing $T = 80$ we have

$$\begin{aligned} 190 - 185e^{-kt} &= 80 \\ 185e^{-kt} &= 110 \\ e^{-kt} &= \frac{110}{185} \\ -kt &= \ln\frac{110}{185} \\ t &= \frac{-1}{k}\ln\frac{110}{185} \\ &= \ln\frac{185}{110}/\ln\frac{185}{161} \\ t &= 3.7414 \text{ hours} = 3 \text{ hours } 44 \text{ mins} \end{aligned}$$

and hence as the turkey was placed in the oven at 9 am, it is cooked at 9am+3 hours 44 mins = 12:44 pm.

Q4.b

Q4.b.i

$7! = 5040$ since john and barbara are as one, so there are 7 objects.

Q4.b.ii

Total number of ways of going through door is 8!, but as Barbara is in front of John in exactly half of these, then the number of ways is $8!/2 = 20160$.

Q4.c

Q4.c.i

Gradient $m_{QO} = \dfrac{aq^2}{2aq} = \dfrac{q}{2}$.

Find the gradient of the tangent at P,

$$y' = \frac{2x}{4a} = \frac{x}{2a} = \frac{2ap}{2a} = p$$

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We are given that QO is perpendicular to the tangent at P and hence $\frac{q}{2}p = -1 \implies pq = -2$.

Q4.c.ii

Gradient, $m_{PO} = \frac{ap^2}{2ap} = \frac{p}{2}$. And $m_{QT} = q$.

So, $m_{PO} \times m_{QT} = \frac{p}{2} \times q = \frac{pq}{2}\frac{-2}{2} = -1$, and hence $PO \perp QT$ and hence $\angle PLQ = 90^o$.

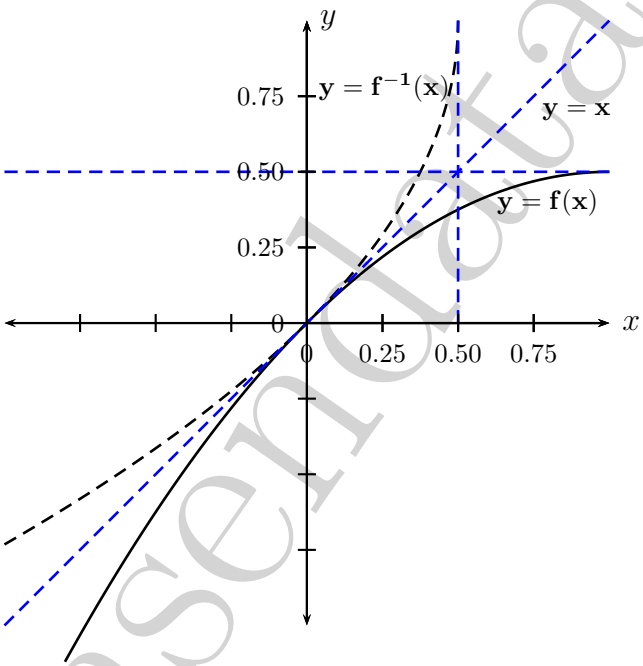
Q4.c.iii

Firstly, $PQLK$ is a cyclic quadrilateral (as $\angle PLQ = \angle PKQ = 90^o$) and PQ is a diameter. Since M is the midpoint of a diameter, then it is the center of the circle. Thus $ML = MK$ as they are both radii of the circle.

Question5

Q5.a

Q5.a.i



Q5.a.ii

Interchange x and y to get the inverse relation: $x = y - y^2/2$. Now make y the subject to find the inverse function;

$x = y - y^2/2$

$y^2 - 2y + 2x = 0$

$y = \frac{2 \pm \sqrt{4 - 4 \times 2x}}{2}$

$y = 1 \pm \sqrt{1 - 2x}$

Hence as we want the lower part of the curve,

$y = 1 - \sqrt{1 - 2x}$

Q5.a.iii

$f^{-1}(3/8) = 1 - \sqrt{1 - 2 \times 3/8} = \frac{1}{2}$

Q5.b

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We may choose $x = a\cos(nt + \alpha)$ as the displacement of a particle in SHM.
Then $\dot{x} = -an\sin(nt + \alpha)$, and $\ddot{x} = -an^2\cos(nt + \alpha)$. Hence $an^2 = 6$ and $an = 2$. Solving these gives $n = 3$ and $a = \dfrac{2}{3}$. Finally the period is $\dfrac{2\pi}{n} = \dfrac{2\pi}{3}$.

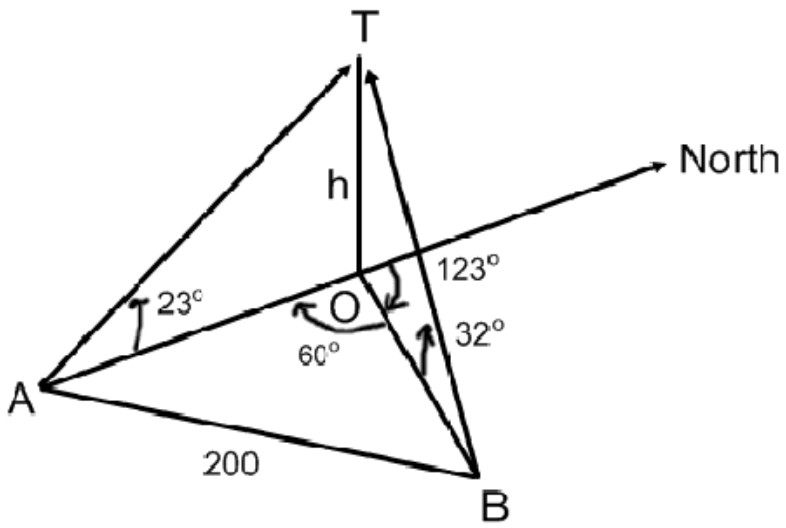
Q5.c

$\angle M = \angle KPQ$ (angle between the tangent and chord is equal to the angle opposite the segment).
$\angle PQM = \angle TPM$ (angle between the tangent and chord is equal to the angle opposite the segment).
$\angle TPM = \angle LPK$ (vertically opposite angles).
$\angle L = \angle PQM$ (angle in a cyclic quad is equal to the exterior opposite angle).
$\therefore \angle L = \angle LPK$. Hence ΔPKL is an isosceles triangle, as required.

Question6

Q6.a

Q6.a.i



Q6.a.ii

Let the height of the tower be h. Then
$\tan 23^o = \dfrac{h}{AO}$ and $\tan 32^o = \dfrac{h}{OB}$, so
$AO = \dfrac{h}{\tan 23^o}$ and $OB = \dfrac{h}{\tan 32^o}$.
Using the cosine rule we have

$$
\begin{aligned}
200^2 &= AO^2 + BO^2 - 2.OA.OB.\cos 60^o \\
200^2 &= AO^2 + BO^2 - OA.OB \\
40000 &= \frac{h^2}{\tan^2 23^o} + \frac{h^2}{\tan^2 32^o} - \frac{h^2}{\tan 23^o \tan 32^o} \\
h &= \frac{200}{\left(\frac{1}{\tan^2 23^o} + \frac{1}{\tan^2 32^o} - \frac{1}{\tan 23^o \tan 32^o}\right)^{1/2}} \\
\therefore h &= 95.99 = 96m \text{ to the nearest meter.}
\end{aligned}
$$

Q6.b

$$
\begin{aligned}
\sin 3\theta + \sin 2\theta &= \sin\theta \\
3\sin\theta - 4\sin^3\theta + 2\sin\theta\cos\theta &= \sin\theta \\
\sin\theta(3 - 4\sin^2\theta + 2\cos\theta - 1) &= 0 \\
\sin\theta(2 - 4(1 - \cos^2\theta) + 2\cos\theta) &= 0 \\
\sin\theta(4\cos^2\theta + 2\cos\theta - 2) &= 0 \\
\sin\theta(2\cos^2\theta + \cos\theta - 1) &= 0 \\
\sin\theta(2\cos\theta - 1)(\cos\theta + 1) &= 0
\end{aligned}
$$

$\sin\theta = 0, \quad 2\cos\theta - 1 = 0, \quad \cos\theta + 1 = 0$

$\theta = n\pi, \quad 2\pi n \pm \cos^{-1}(0.5), \quad 2\pi n \pm \cos^{-1}(-1)$

$\theta = n\pi, \quad 2\pi n \pm \frac{\pi}{3}, \quad 2\pi n \pm \pi$

where n is an integer.

Hence the final result in the required range, $0 \le \theta \le 2\pi$, is $\theta = 0, \frac{\pi}{3}, \pi, \frac{5\pi}{3}, 2\pi$.

Q6.c

Q6.c.i

Expanding the binomial we have,

$x^q(1+x)^{p+q} = x^q \sum_{r=0}^{p+q} \binom{p+q}{r} x^r = \sum_{r=0}^{p+q} \binom{p+q}{r} x^{r-q}$. The term independent of x has index $r - q = 0 \implies r = q$. Hence the term independent of x is $\binom{p+q}{q}$.

Q6.c.ii

$$(1+x)^p = \sum_{r=0}^{p} \binom{p}{r} x^r$$

$$(1+x^{-1})^q = \sum_{s=0}^{q} \binom{q}{s} x^{-s}$$

$$\therefore (1+x)^p (1+x^{-1})^q = \sum_{r=0}^{p}\sum_{s=0}^{q} \binom{p}{r}\binom{q}{s} x^{r-s}$$

Terms independent of x occur when $r = s$ so we combine all those terms

$$= \binom{p}{0}\binom{q}{0} + \binom{p}{1}\binom{q}{1} + \cdots + \binom{p}{p}\binom{q}{p}$$

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where the last term is as it is as $p \leq q$, so it stops there.
Hence equate this expression to that in part (i) and we have,
$\binom{p}{0}\binom{q}{0} + \binom{p}{1}\binom{q}{1} + \cdots + \binom{p}{p}\binom{q}{p} = \binom{p+q}{q}$.

Question7

Q7.a

We have $Vt\sin\theta - gt^2/2 = h$
$gt^2/2 - Vt\sin\theta + h = 0.$

Sum Roots: $t_1 + t_2 = -B/A = \dfrac{V\sin\theta}{g/2} = \dfrac{2V\sin\theta}{g}$

Product of roots: $t_1\,t_2 = \dfrac{h}{g/2} = \dfrac{2h}{g}$

as required.

Q7.b

$$\begin{aligned}
\tan\alpha + \tan\beta &= \frac{h}{Vt_1\cos\theta} + \frac{h}{Vt_2\cos\theta} \\
&= \frac{h}{V\cos\theta}\left(\frac{1}{t_1} + \frac{1}{t_2}\right) \\
&= \frac{h}{V\cos\theta}\left(\frac{t_1+t_2}{t_1t_2}\right) \\
&= \frac{h}{V\cos\theta} \times \frac{2V\sin\theta}{g} \times \frac{g}{2h} \\
&= \frac{\sin\theta}{\cos\theta} \\
&= \tan\theta
\end{aligned}$$

Q7.c

$$\begin{aligned}
\tan\alpha\,\tan\beta &= \frac{h}{Vt_1\cos\theta} \times \frac{h}{Vt_2\cos\theta} \\
&= \frac{h^2}{V^2\cos^2\theta\, t_1t_2} \\
&= \frac{h^2 g}{V^2\cos^2\theta\, 2h} \\
&= \frac{gh}{2V^2\cos^2\theta}
\end{aligned}$$

as required.

Q7.d

From the graph we see that
$\tan\alpha = \frac{h}{OP} \implies \cot\alpha = \frac{OP}{h} \implies OP = h\cot\alpha$ and
$\tan\beta = \frac{h}{PN} \implies \cot\beta = \frac{PN}{h} \implies PN = h\cot\beta.$
$r = OP + PN$
$r = h\cot\alpha + h\cot\beta$
$r = h(\cot\alpha + \cot\beta)$

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$w = r - OP - QN$
but $OP = QN$ by symmetry as the projectile is fired on level ground.
$\therefore w = r - 2OP$
$w = h\cot\alpha + h\cot\beta - 2h\cot\alpha$
$w = h(\cot\beta - \cot\alpha)$
as required.

Q7.e

Firstly from the equations given in the question,
$\dot{x} = V\cos\theta$
$\dot{y} = V\sin\theta - gt$
Hence,

$$\begin{aligned}
\tan\phi &= \frac{\dot{y}}{\dot{x}} \\
&= \frac{V\sin\theta - gt}{V\cos\theta} \\
&= \tan\theta - \frac{gt}{V\cos\theta} \\
&= \tan\theta - \frac{g}{V\cos\theta}\left(\frac{V\sin\theta}{g} - \frac{\sqrt{V^2\sin^2\theta - 2hg}}{g}\right) \\
\tan\phi &= \frac{\sqrt{V^2\sin^2\theta - 2hg}}{V\cos\theta} \\
V^2\tan^2\phi\cos^2\theta &= V^2\sin^2\theta - 4V^2\cos^2\theta\tan\alpha\,\tan\beta \\
\tan^2\phi &= \tan^2\theta - 4\tan\alpha\,\tan\beta \\
&= \tan^2\alpha + \tan^2\beta + 2\tan\alpha\,\tan\beta - 4\tan\alpha\,\tan\beta \\
&= \tan^2\alpha + \tan^2\beta - 2\tan\alpha\,\tan\beta \\
\tan^2\phi &= (\tan\alpha - \tan\beta)^2 \\
\therefore \tan\phi &= \tan\alpha - \tan\beta
\end{aligned}$$

taking the positive square root since $\alpha > \beta$ and $\phi < \pi/2$
as required.

Q7.f

Using part (d)and (e) we have

$$\begin{aligned}
LHS &= \frac{w}{\tan\phi} \\
&= \frac{h(\cot\beta - \cot\alpha)}{\tan\phi} \\
&= \frac{h(\cot\beta - \cot\alpha)}{\tan\alpha - \tan\beta} \\
&= \frac{h\left(\dfrac{\tan\alpha - \tan\beta}{\tan\alpha\tan\beta}\right)}{\tan\alpha - \tan\beta} \\
&= h\cot\alpha\cot\beta
\end{aligned}$$

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$$RHS = \frac{r}{\tan\theta}$$
$$= \frac{h(\cot\alpha + \cot\beta)}{\tan\theta}$$
$$= \frac{h\left(\dfrac{\tan\alpha + \tan\beta}{\tan\alpha\tan\beta}\right)}{\tan\alpha + \tan\beta}$$
$$= h\cot\alpha\cot\beta = LHS$$

as required.

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