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NSW BOS Mathematics Extension 1 Solutions 2006
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Q1a $\int \frac{dx}{49+x^2} = \frac{1}{7}\int \frac{7}{7^2+x^2}\,dx = \frac{1}{7}\tan^{-1}\left(\frac{x}{7}\right)+c$.

Q1b $u = x^4+8$, $\frac{du}{dx} = 4x^3$.

$$\int x^3\sqrt{x^4+8}\,dx = \int \frac{1}{4}\sqrt{u}\,\frac{du}{dx}\,dx = \int \frac{1}{4}u^{\frac{1}{2}}du = \frac{u^{\frac{3}{2}}}{6}+c = \frac{\left(x^4+8\right)^{\frac{3}{2}}}{6}+c$$

Q1c $\lim_{x\to 0}\frac{\sin 5x}{3x} = \lim_{x\to 0}\frac{5}{3}\times\frac{\sin 5x}{5x} = \frac{5}{3}\lim_{x\to 0}\frac{\sin 5x}{5x} = \frac{5}{3}$.

Q1d

$$\frac{\sin^3\theta+\cos^3\theta}{\sin\theta+\cos\theta}-1 = \frac{(\sin\theta+\cos\theta)(\sin^2\theta-\sin\theta\cos\theta+\cos^2\theta)}{(\sin\theta+\cos\theta)}-1$$
$$= \left(\sin^2\theta-\sin\theta\cos\theta+\cos^2\theta\right)-1 = \left(1-\sin\theta\cos\theta\right)-1$$
$$= -\sin\theta\cos\theta = -\frac{1}{2}\sin 2\theta\,.$$

Q1e $y = x^3$, $\frac{dy}{dx} = 3x^2 = 12$, $\therefore x = \pm 2$ and $y = \pm 8$.

$y = 12x+b$, $8 = 12(2)+b$, $b = -16$.

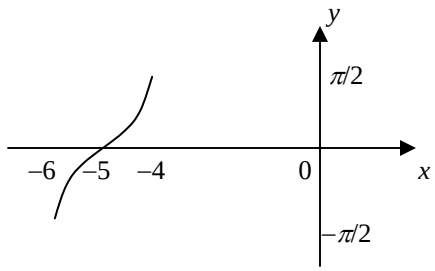
$-8 = 12(-2)+b$, $b = 16$.

Q2ai $f(x) = \sin^{-1}(x+5)$ is defined for $-1 \le x+5 \le 1$,

i.e. $-6 \le x \le -4$. Domain: $[-6,-4]$, range: $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$.

Q2aii $f'(x) = \frac{1}{\sqrt{1-(x+5)^2}}$, $f'(-5) = \frac{1}{\sqrt{1-(-5+5)^2}} = 1$.

Q2aiii



Q2bi $(1+x)^n = 1+\binom{n}{1}x+\binom{n}{2}x^2+.....+\binom{n}{r}x^r+.....+\binom{n}{n}x^n$,

$$\frac{d}{dx}(1+x)^n = \frac{d}{dx}\left[1+\binom{n}{1}x+\binom{n}{2}x^2+.....+\binom{n}{r}x^r+.....+\binom{n}{n}x^n\right]$$
$$n(1+x)^{n-1} = \binom{n}{1}+2\binom{n}{2}x+.....+r\binom{n}{r}x^{r-1}+.....+n\binom{n}{n}x^{n-1}.$$

Q2bii Let $x = 2$,

$$n3^{n-1} = \binom{n}{1}+.....+r\binom{n}{r}2^{r-1}+.....+n\binom{n}{n}2^{n-1} = \sum_{r=1}^{n} r\binom{n}{r}2^{r-1}.$$

Q2ci Given $y = \frac{1}{2}(p+r)x-apr$. At point U, $x = 0$,

$\therefore y = -apr$, hence $U(0,-apr)$.

Q2cii Given the tangent at $P(2ap,ap^2)$ is $y = px-ap^2$, by symmetry the tangent at $Q(2aq,aq^2)$ is $y = qx-aq^2$.

At point T, $px-ap^2 = qx-aq^2$, $\therefore px-qx = ap^2-aq^2$,

$(p-q)x = a(p-q)(p+q)$, $\therefore x = a(p+q)$, where $p \ne q$.

$\therefore y = px-ap^2 = pa(p+q)-ap^2 = apq$.

Hence $T(a(p+q),apq)$.

Q2ciii Since $Q(2aq,aq^2)$ and $R(2ar,ar^2)$ are two different points having the same *y*-coordinate, $\therefore q^2 = r^2$ and $q \ne r$,

$\therefore q = -r$.

$\therefore U(0,-apr)$ and $T(a(p+q),apq)$ are two different points having the same *y*-coordinate, $\therefore TU$ is a horizontal line perpendicular to the vertical axis of the parabola.

Q3a $\int_0^{\frac{\pi}{4}} \sin^2 x\,dx = \int_0^{\frac{\pi}{4}} \frac{1}{2}(1-\cos 2x)\,dx = \left[\frac{1}{2}\left(x-\frac{\sin 2x}{2}\right)\right]_0^{\frac{\pi}{4}}$

$$= \frac{1}{2}\left(\frac{\pi}{4}-\frac{1}{2}\right) = \frac{\pi-2}{8}.$$

Q3bi $f(x) = 3\log_e x - x$, $f(1.5) = 3\log_e 1.5 - 1.5 = -0.2836$,

$f(2) = 3\log_e 2 - 2 = 0.0794$. $\therefore f(x)$ has a zero at $1.5 < x < 2$,

$\therefore x$-coordinate of P is $1.5 < x < 2$.

Q3bii $f'(x) = \frac{3}{x}-1$, $x_1 = 1.5$, $f'(1.5) = \frac{3}{1.5}-1 = 1$.

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 1.5 - \frac{-0.2836}{1} = 1.78.$$

Q3ci Three blocks high: $5\times 4\times 3 = 60$

Q3cii Total number of 2, 3, 4 and 5-block towers:
$5\times 4+5\times 4\times 3+5\times 4\times 3\times 2+5\times 4\times 3\times 2\times 1 = 320$

Q3di Since $\angle QKT+\angle QMT = 180^0$, $\therefore QKTM$ is a cyclic quadrilateral.

Q3dii $\angle KMT$ and $\angle KQT$ are subtended by the same arc KT, $\therefore$ they are equal.

Q3diii $\angle PTN = \angle PQT$ (The angle between a chord and a tangent at an end of the chord equals the angle in the alternate segment).

$\angle PTN = \angle PQT = \angle KQT = \angle KMT$, i.e. $\angle PTN$ and $\angle KMT$ are equal corresponding angles, $\therefore MK \,//\, TP$.

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Q4ai and Q4aii $P(x) = x^3 + rx^2 + sx + t = (x-1)(x-\alpha)(x+\alpha)$

$= x^3 - x^2 - \alpha^2 x + \alpha^2$.

Comparing coefficients: $r = -1$, $s = -\alpha^2$, $t = \alpha^2$.

$\therefore s + t = 0$.

Q4bi $T = \dfrac{2\pi}{n}$, $5 = \dfrac{2\pi}{n}$, $\therefore n = \dfrac{2\pi}{5}$. $\therefore x = 18\cos\left(\dfrac{2\pi}{5}t\right)$.

Q4bii A rest position is at one of the extreme positions.

First time at $x = 9$, $9 = 18\cos\left(\dfrac{2\pi}{5}t\right)$, $\therefore \cos\left(\dfrac{2\pi}{5}t\right) = 0.5$,

$\dfrac{2\pi}{5}t = \dfrac{\pi}{3}$, $t = \dfrac{5}{6}$ s. Required time is $\dfrac{5}{6}$ s.

Q4ci $\ddot{x} = 18x^3 + 27x^2 + 9x$. At $t = 0$, $x = -2$, $v = -6$.

$\dfrac{d\left(\frac{1}{2}v^2\right)}{dx} = 18x^3 + 27x^2 + 9x$,

$\dfrac{1}{2}v^2 = \int \left(18x^3 + 27x^2 + 9x\right)dx$, $\dfrac{1}{2}v^2 = \dfrac{9x^4}{2} + 9x^3 + \dfrac{9x^2}{2} + d$,

$\therefore v^2 = 9x^4 + 18x^3 + 9x^2 + c$.

Using $x = -2$, $v = -6$, c is 0.

$\therefore v^2 = 9x^4 + 18x^3 + 9x^2 = 9x^2\left(x^2 + 2x + 1\right) = 9x^2(x+1)^2$.

Q4cii At $t = 0$, $x = -2$, $v = -6$, $\ddot{x} = -54$, $\therefore v$ remains negative and $x < -2$ as time progresses.

$\therefore v = -\sqrt{9x^2(x+1)^2} = -3x(1+x)$, i.e. $\dfrac{dx}{dt} = -3x(1+x)$.

Hence $\dfrac{dt}{dx} = -\dfrac{1}{3} \times \dfrac{1}{x(1+x)}$, $t = -\dfrac{1}{3}\int \dfrac{1}{x(1+x)}dx$,

$\therefore \int \dfrac{1}{x(1+x)}dx = -3t$.

Q4ciii $\log_e\left(1 + \dfrac{1}{x}\right) = 3t + c$. At $t = 0$, $x = -2$,

$\therefore \log_e\left(1 - \dfrac{1}{2}\right) = c$, $c = \log_e \dfrac{1}{2} = -\log_e 2$.

$\therefore \log_e\left(1 + \dfrac{1}{x}\right) = 3t - \log_e 2$, $\log_e\left(1 + \dfrac{1}{x}\right) + \log_e 2 = 3t$,

$\therefore \log_e 2\left(1 + \dfrac{1}{x}\right) = 3t$, $2\left(1 + \dfrac{1}{x}\right) = e^{3t}$, $1 + \dfrac{1}{x} = 0.5e^{3t}$,

$\dfrac{1}{x} = 0.5e^{3t} - 1$, $x = \dfrac{1}{0.5e^{3t} - 1}$.

Q5a $y = 10e^{-0.7t} + 3$, $\dfrac{dy}{dt} = -0.7\left(10e^{-0.7t}\right) = -0.7(y-3)$.

Q5b All functions (relations) have an inverse. Does the question mean to show that $f(x) = \log_e\left(1 + e^x\right)$ has an inverse **function**?

$f'(x) = \dfrac{e^x}{1+e^x} > 0$ for all x, $\therefore f(x)$ is an increasing function for all x, $\therefore f(x)$ is a one-to-one function and hence $f^{-1}(x)$ exists.

Q5ci $V = \dfrac{\pi}{3}x^2(3r - x) = \pi r x^2 - \dfrac{\pi}{3}x^3$, $\dfrac{dV}{dx} = 2\pi r x - \pi x^2$.

$\dfrac{dV}{dt} = \dfrac{dV}{dx} \times \dfrac{dx}{dt}$, $\therefore \dfrac{dx}{dt} = \dfrac{\frac{dV}{dt}}{\frac{dV}{dx}} = \dfrac{k}{2\pi r x - \pi x^2} = \dfrac{k}{\pi x(2r - x)}$.

Q5cii Since the rate is constant, $V = kt$, $t = \dfrac{V}{k}$.

When $x = \dfrac{1}{3}r$, $t_i = \dfrac{8\pi}{81k}r^3$.

When $x = \dfrac{2}{3}r$, $t_f = \dfrac{28\pi}{81k}r^3$.

$\therefore \dfrac{t_f}{t_i} = \dfrac{28}{8} = 3.5$

Q5di $\tan(\alpha - \beta) = \dfrac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta}$,

$\therefore 1 + \tan\alpha\tan\beta = \dfrac{\tan\alpha - \tan\beta}{\tan(\alpha - \beta)}$.

Let $\alpha = (n+1)\theta$ and $\beta = n\theta$.

$1 + \tan(n+1)\theta\tan n\theta = \dfrac{\tan(n+1)\theta - \tan n\theta}{\tan\theta}$,

$\therefore 1 + \tan(n+1)\theta\tan n\theta = \cot\theta\left(\tan(n+1)\theta - \tan n\theta\right)$,

$\therefore 1 + \tan n\theta\tan(n+1)\theta = \cot\theta\left(\tan(n+1)\theta - \tan n\theta\right)$.

Q5dii Given $1 + \tan n\theta\tan(n+1)\theta = \cot\theta\left(\tan(n+1)\theta - \tan n\theta\right)$, prove by induction that for $n \geq 1$,

$\tan\theta\tan 2\theta + \tan 2\theta\tan 3\theta + \ldots + \tan n\theta\tan(n+1)\theta$

$= -(n+1) + \cot\theta\tan(n+1)\theta$

It is true for $n = 1$: $1 + \tan\theta\tan 2\theta = \cot\theta\left(\tan 2\theta - \tan\theta\right)$,

$1 + \tan\theta\tan 2\theta = \cot\theta\tan 2\theta - \cot\theta\tan\theta$,

$\therefore \tan\theta\tan 2\theta = -2 + \cot\theta\tan 2\theta$.

Assume that it is true for $n = k$, it is also true for $n = k+1$:

$\tan\theta\tan 2\theta + \tan 2\theta\tan 3\theta + \ldots + \tan k\theta\tan(k+1)\theta$

$= -(k+1) + \cot\theta\tan(k+1)\theta$

then $\tan\theta\tan 2\theta + \tan 2\theta\tan 3\theta + \ldots + \tan k\theta\tan(k+1)\theta + \tan(k+1)\theta\tan(k+2)\theta$

$= -(k+1) + \cot\theta\tan(k+1)\theta + \tan(k+1)\theta\tan(k+2)\theta$

$= -(k+1) + \cot\theta\tan(k+1)\theta + \cot\theta\left(\tan(k+2)\theta - \tan(k+1)\theta\right) - 1$

$= -\left((k+1)+1\right) + \cot\theta\tan\left((k+1)+1\right)\theta$.

$\therefore$ it is true for all $n \geq 1$.

Q6ai $L^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$

$= (a - Vt\cos\theta)^2 + (Vt - Vt\sin\theta)^2$

$= a^2 - 2aVt\cos\theta + V^2t^2\cos^2\theta + V^2t^2 - 2V^2t^2\sin\theta + V^2t^2\sin^2\theta$

$= V^2t^2\sin^2\theta + V^2t^2\cos^2\theta + V^2t^2 - 2V^2t^2\sin\theta - 2aVt\cos\theta + a^2$

$= V^2t^2\left(\sin^2\theta + \cos^2\theta\right) + V^2t^2 - 2V^2t^2\sin\theta - 2aVt\cos\theta + a^2$

$= 2V^2t^2(1 - \sin\theta) - 2aVt\cos\theta + a^2$

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Q6aii Smallest distance occurs at where $\frac{d(L^2)}{dt}=0$,

$4V^2(1-\sin\theta)t-2aV\cos\theta=0$, $\therefore t=\frac{a\cos\theta}{2V(1-\sin\theta)}$.

$L=\sqrt{2V^2t^2(1-\sin\theta)-2aVt\cos\theta+a^2}$,

$$L_{\min}=\sqrt{2V^2\left(\frac{a\cos\theta}{2V(1-\sin\theta)}\right)^2(1-\sin\theta)-2aV\frac{a\cos\theta}{2V(1-\sin\theta)}\cos\theta+a^2}$$

$$=\sqrt{\frac{a^2\cos^2\theta}{2(1-\sin\theta)}-\frac{a^2\cos^2\theta}{(1-\sin\theta)}+a^2}$$

$$=a\sqrt{1-\frac{\cos^2\theta}{2(1-\sin\theta)}}=a\sqrt{1-\frac{1-\sin^2\theta}{2(1-\sin\theta)}}$$

$$=a\sqrt{1-\frac{(1-\sin\theta)(1+\sin\theta)}{2(1-\sin\theta)}}=a\sqrt{1-\frac{1+\sin\theta}{2}}=a\sqrt{\frac{1-\sin\theta}{2}}.$$

Q6aiii Particle 1 is ascending when its $\frac{dy}{dt}>0$,

i.e. $V\sin\theta-gt>0$, $t<\frac{V\sin\theta}{g}$.

$\therefore \frac{a\cos\theta}{2V(1-\sin\theta)}<\frac{V\sin\theta}{g}$, $\therefore V^2>\frac{ag\cos\theta}{2\sin\theta(1-\sin\theta)}$,

$\therefore V>\sqrt{\frac{ag\cos\theta}{2\sin\theta(1-\sin\theta)}}$.

Q6bi Four-member team: Let Y be the number of competitors not completing the course.

$$\Pr(Y\geq 3)=\Pr(Y=3)+\Pr(Y=4)=\binom{4}{3}pq^3+q^4=4pq^3+q^4.$$

Q6bii Four-member team. Let X be the number of competitors completing the course.

$$\Pr(X\geq 2)=\binom{4}{2}p^2q^2+\binom{4}{3}p^3q+p^4$$

$$=6(1-q)^2q^2+4(1-q)^3q+(1-q)^4$$

$$=(1-q)^2\left(6q^2+4q(1-q)+(1-q)^2\right)$$

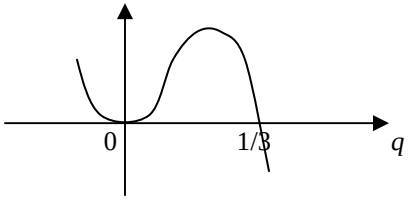
$$=(1-q)^2\left(1+2q+3q^2\right).$$

Q6biii Two-member team: $\Pr(X\geq 1)=1-\Pr(X=0)=1-q^2$.

Q6biv $1-q^2>(1-q)^2(1+2q+3q^2)$,

$(1-q)(1+q)>(1-q)^2(1+2q+3q^2)$,

$1+q>(1-q)(1+2q+3q^2)$, $\therefore q^2-3q^3<0$, $q^2(1-3q)<0$.



$q^2(1-3q)<0$ when $q>\frac{1}{3}$, $\therefore \frac{1}{3}<q<1$.

Q7a $A=\frac{1}{2}r^2(2\theta)-\frac{1}{2}r^2\sin 2\theta=r^2\theta-\frac{1}{2}r^2(2\sin\theta\cos\theta)$

$=r^2(\theta-\sin\theta\cos\theta)$.

Q7b $w=r(2\theta)$, $r=\frac{w}{2\theta}$, $\therefore A=\frac{w^2}{4\theta^2}(\theta-\sin\theta\cos\theta)$.

$$\frac{dA}{d\theta}=-\frac{2w^2}{4\theta^3}(\theta-\sin\theta\cos\theta)+\frac{w^2}{4\theta^2}(1-\cos^2\theta+\sin^2\theta)$$

$$=-\frac{2w^2}{4\theta^3}(\theta-\sin\theta\cos\theta)+\frac{w^2}{4\theta^2}(1-\cos^2\theta+1-\cos^2\theta)$$

$$=-\frac{2w^2}{4\theta^3}(\theta-\sin\theta\cos\theta)+\frac{2w^2}{4\theta^2}(1-\cos^2\theta)$$

$$=\frac{w^2}{2\theta^3}(\sin\theta\cos\theta-\theta\cos^2\theta)$$

$$=\frac{w^2\cos\theta(\sin\theta-\theta\cos\theta)}{2\theta^3}.$$

Q7c Let $g(\theta)=\sin\theta-\theta\cos\theta$,

$g'(\theta)=\cos\theta-\cos\theta+\theta\sin\theta=\theta\sin\theta$.

For $0<\theta<\pi$, $g'(\theta)=\theta\sin\theta>0$,

$\therefore g(\theta)=\sin\theta-\theta\cos\theta$ is an increasing function in $0<\theta<\pi$,

and $g(0)=0$, hence $g(\theta)=\sin\theta-\theta\cos\theta>0$ for $0<\theta<\pi$.

Q7d $\frac{dA}{d\theta}=0$, $\frac{w^2\cos\theta(\sin\theta-\theta\cos\theta)}{2\theta^3}=0$, where $0<\theta<\pi$.

Since $w>0$ and $\sin\theta-\theta\cos\theta>0$,

$\therefore \cos\theta=0$ and $\theta=\frac{\pi}{2}$ is the only solution in $0<\theta<\pi$.

Q7e

θ	1	$\frac{\pi}{2}$	2
$\frac{dA}{d\theta}$	+	0	–

$\therefore A$ is maximum when $\theta=\frac{\pi}{2}$.

$\therefore A=\frac{w^2}{4\theta^2}(\theta-\sin\theta\cos\theta)$,

$A_{\max}=\frac{w^2}{4\left(\frac{\pi}{2}\right)^2}\left(\frac{\pi}{2}-\sin\frac{\pi}{2}\cos\frac{\pi}{2}\right)=\frac{w^2}{2\pi}$ square units.

Please inform mathline@itute.com re conceptual, mathematical and/or typing errors.

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