HSC 2005 MATHEMATICS EXTENSION 1 (3 unit) EXAM : ANSWERS/SOLUTIONS.

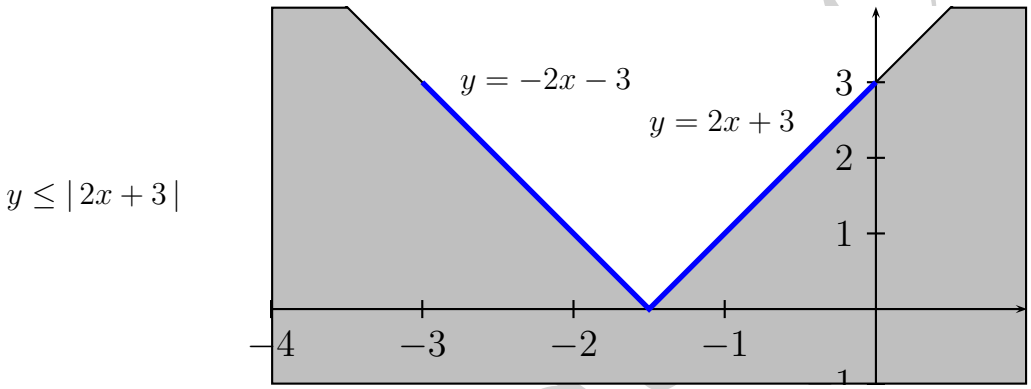
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Question 1

Q1.a

$$\int \frac{1}{x^2+49}\,dx = \int \frac{1}{x^2+7^2}\,dx = \frac{1}{7}\tan^{-1}\frac{x}{7} + c$$

Q1.b



Q1.c

We have $-1 \leq \frac{x}{4} \leq 1$ since the domain of $y = \cos^{-1}x$ is $-1 \leq x \leq 1$. Hence the domain of $y = \cos^{-1}\frac{x}{4}$ is $-4 \leq x \leq 4$.

The corresponding range is $0 \leq y \leq \pi$, that is it inherits the full range since as x varies from -4 to 4, the $x/4$ varies from -1 to 1.

Q1.d

$u = 2x^2+1$, $du = 4x\,dx$

$$\int x\left(2x^2+1\right)^{5/4}\,dx = \frac{1}{4}\int u^{5/4}\,du = \frac{1}{4}.\frac{4}{9}u^{9/4} + c = \frac{1}{9}\left(2x^2+1\right)^{9/4} + c$$

Q1.e

The line joining the points (x_1, y_1) and (x_2, y_2) is divided internally in the ratio $m : n$ by the point $\left(\frac{m\,x_2 + n\,x_1}{m+n}, \frac{m\,y_2 + n\,y_1}{m+n}\right)$.

Hence

$$(1,4) = \left(\frac{2\,x + 3\,.(-1)}{2+3}, \frac{2\,y + 3\,.8}{2+3}\right)$$

Therefore, $\frac{2x-3}{5} = 1$ and $\frac{2y+24}{5} = 4$. Solving these we btain $x = 4, y = -2$.

That is $\mathbf{B(4,-2)}$.

Q1.f

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We have the two lines, $y = 3x + 5$ and $y = mx + 4$, so $m_1 = 3, m_2 = m$ we use our angle formula,

$$\tan 45^o = \left|\frac{m_1 + m_2}{1 - m_1\, m_2}\right| = \left|\frac{3+m}{1-3\,m}\right|$$
$$1 = \left|\frac{3+m}{1-3\,m}\right|$$
$$|1 - 3\,m| = |m + 3|$$
$$|1 - 3\,m|^2 = |m + 3|^2$$
$$(1 - 3\,m)^2 = (m + 3)^2$$
$$1 + 9m^2 - 6m = m^2 + 9 + 6m$$
$$8m^2 - 12m - 8 = 0$$
$$2m^2 - 3m - 2 = 0$$
$$(2m + 1)(m - 2) = 0$$
$$\mathbf{m = -\tfrac{1}{2},\ 2}$$

Alternatively , we could also proceed by considering the two cases as follows,

$\lvert 1 - 3\,m\rvert = \lvert m + 3\rvert$	
$(1 - 3\,m) = \pm(m + 3)$	
$1 - 3m = m + 3$	$1 - 3m = -(m + 3)$
$1 - 3 = m + 3m$	$1 - 3m = -m - 3$
$-2 = 4m$	$1 + 3 = -m + 3m$
$\mathbf{m = -\tfrac{1}{2}}$	$4 = 2m$
	$\mathbf{m = 2}$

giving the same solution as it should.

Question 2

Q2.a

$$\frac{d}{dx}\left(2\sin^{-1}5x\right) = 2.\frac{1}{\sqrt{1-25x^2}} \times 5 = \frac{10}{\sqrt{1-25x^2}}.$$

Q2.b Expanding we have,

$$\left(2x - \frac{1}{x^2}\right)^{12} = \sum_{r=0}^{12}\binom{12}{r}(2x)^{12-r}(-1)^r x^{-2r} = \sum_{r=0}^{12}\binom{12}{r}(-1)^r 2^{12-r}x^{12-3r}.$$

The term independant of x is the constant term which corresponds to $12 - 3r = 0$ and hence $\mathbf{r = 4}$.

The constant term is thus $\binom{12}{4}.(-1)^4.2^{12-4} = \binom{12}{4}.2^8 = \mathbf{126,720}$.

Q2.c

Q2.c.i

$$\begin{aligned}\frac{d}{dx}\left(e^{3x}(\cos x - 3\sin x)\right) &= e^{3x}(-\sin x - 3\cos x) + 3e^{3x}(\cos x - 3\sin x)\\ &= e^{3x}(-10\sin x - 3\cos x + 3\cos x) = \mathbf{-10e^{3x}}\ \sin\mathbf{x}\end{aligned}$$

Q2.c.ii

Hence $\displaystyle\int e^{3x}\sin x\,dx = -\frac{1}{10}\int -10e^{3x}\sin x\,dx = -\frac{\mathbf{1}}{\mathbf{10}}\mathbf{e^{3x}}(\cos\mathbf{x} - \mathbf{3}\sin\mathbf{x}).$

Q2.d

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We are given that when $t = 0$, $T = 25°$.

Q2.d.i

$$\begin{aligned} \text{LHS} &= \frac{dT}{dt} = \frac{d}{dt}\left(3 + Ae^{-kt}\right) \\ &= 0 + -kAe^{-kt} \\ &= -k\left(Ae^{-kt}\right) \\ &= -k(T-3) \quad \text{since } T = 3 + Ae^{-kt} \\ &= \text{RHS} \end{aligned}$$

and hence we have shown that it satisfies the equation. **Q2.d.ii**

When $t = 10$, $T = 11°$.

$\therefore\ 25 = 3 + Ae^0$ and so $A = 22$ and so we have $T = 3 + 22e^{-kt}$. We now find k using the other condition,

$$\begin{aligned} 11 &= 3 + 22e^{-10k} \\ 22e^{-10k} &= 9 \\ e^{-10k} &= 9/22 \\ -10k &= \ln \tfrac{9}{22} \\ 10k &= \ln \tfrac{22}{9} \\ k &= \tfrac{1}{10} \ln \tfrac{22}{9} \end{aligned}$$

Question 3

Q3.a

Q3.a.i $g(x) = x^2 - \ln(x+1)$

$g(0.7) = -0.04 < 0$ and $g(0.9) = 0.168 > 0$.

Hence as the function is continuous for all $0.7 \le x \le 0.9$ then it must cross the x-axis in between somewhere.

Q3.a.ii

We use a signs table as follows

x	$f(x)$	Deduction so far
0.7	$-$	
0.9	$+$	$0.7 < x < 0.9$
0.8	$+$	$0.7 < x < \mathbf{0.8}$
0.75	$+$	$0.7 < x < \mathbf{0.75}$

Hence to one decimal place the root is $\mathbf{x_* = 0.7}$

[The last line in the table is enough as this shows that a root definitely lies **between** $x = 0.7$ and $x = 0.75$ but not equal to 0.75. Any value of x with $0.7 < x < 0.75$ is equal to 0.7 when rounded to one decimal place.]

Further: Continuing this process and not necessarily halfing the interval each time we can show that since $f(a) < 0$, $f(b) > 0$ where $a = 0.7468815$, $b = 0.7468820$ then the root satisfies $a < x_* < b$ so that to six decimal places the roots is

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$x_* = 0.746882$. Out of interest lets calculate the function value here, it is $f(0.746882) = 000000237$ which correct to six decimal places equals zero.

Q3.b

Q3.b.i Recall that $\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \mp \sin\beta\cos\alpha$. So, LHS $= \sin 5x\cos 4x + \sin 4x\cos 5x + \sin 5x\cos 4x - \sin 4x\cos 5x = 2\sin 5x\cos 4x$ as required.

Q3.b.ii $\sin 5x\cos 4x\,dx = \dfrac{1}{2}\displaystyle\int(\sin 9x + \sin x)\,dx$

$= -\dfrac{1}{2}\left[-\dfrac{\cos 9x}{9} - \cos x\right] + c = -\dfrac{1}{18}\cos 9x - \dfrac{1}{2}\cos x + c$

Q3.c

$$\begin{aligned} f'(x) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} \\ &= \frac{\left[(x+h)^2+5(x+h)\right]-(x^2+5x)}{h} \\ &= \frac{x^2+2xh+h^2+5x+5h-x^2-5x}{h} \\ &= \frac{2xh+h^2+5h}{h} \\ &= \frac{h(2x+h+5)}{h} \\ &= 2x+h+5 \\ f'(x) &= 2x+5 \end{aligned}$$

Q3.d

Q3.d.i

$EB = 7-4 = 3$, $ED = x$, $CD = l$

Recall the circle result $AE.EB = CE.ED$

$\therefore 4.3 = (l-x)x = lx - x^2$

which simplifies to give $x^2 - lx + 12 = 0$, as required.

Q3.d.ii Starting with $x^2 - lx + 12 = 0$, make l the subject;

$l = x + 12\,x^{-1}$.
$l' = 1 - 12x^{-2} = 0$ (for stat pts.)
$12/x^2 = 1$ and so
$x = +2\sqrt{3}$ and since $l''(2\sqrt{3}) = 24y^{-3} > 0$ we have shown the minimum length occurs at this x value.

$$l(2\sqrt{3}) = 2\sqrt{3} + \frac{12}{2\sqrt{3}} = 2\sqrt{3} + \frac{6}{\sqrt{3}}.\frac{\sqrt{3}}{\sqrt{3}} = 2\sqrt{3} + 2\sqrt{3} = \mathbf{4\sqrt{3}}.$$

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The length of the shortest chord through E is $\mathbf{4\sqrt{3}}$ units.

Question 4

Q4.a

Using the substituton $u = \sin x$, $dx = \cos x\,dx$ and noting that when $x = 0, u = \sin 0 = 0$ and when $x = \pi/4, u = \sin \pi/4 = 1/\sqrt{2}$ we have

$$\int_0^{\pi/4} \cos x \sin^2 x\,dx = \int_0^{1/\sqrt{2}} u^2\,du = \left[\frac{u^3}{3}\right]_0^{1/\sqrt{2}} = \frac{1}{3\,.\,2^{3/2}} = \frac{1}{3\,.\,2\,.\,\sqrt{2}} = \mathbf{\frac{1}{6\sqrt{2}}}$$

Q4.b

$$\begin{aligned} \operatorname{cosec}\theta + \cot\theta &= \frac{1}{\sin\theta} + \frac{\cos\theta}{\sin\theta} \\ &= \frac{1+\cos\theta}{\sin\theta} \\ &= \frac{2-2\sin^2\frac{\theta}{2}}{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}} \end{aligned} \quad \Bigg| \quad \begin{aligned} &= \frac{2(1-\sin^2\frac{\theta}{2})}{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}} \\ &= \frac{\cos^2\frac{\theta}{2}}{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}} \\ &= \frac{\cos\frac{\theta}{2}}{2\sin\frac{\theta}{2}} \\ &= \cot\frac{\theta}{2} \end{aligned}$$

as required.

Q4.c

Q4.c.i

We solve simultaneously, assuming that $p \neq q$ (for otherwise the question makes no sense)

$$\begin{aligned} x + py &= 2ap + ap^3 \quad (1) \\ x + qy &= 2aq + aq^3 \quad (2) \\ \therefore\quad (p-q)y &= 2a(p-q) + a(p^3 - q^3) \\ (p-q)y &= 2a(p-q) + a(p-q)(p^2+pq+q^2) \\ \therefore\quad y &= 2a + a(p^2+pq+q^2) \\ y &= a[p^2+pq+q^2+2] \\ \therefore\quad x &= 2ap + ap^3 - pa(p^2+pq+q^2+2) \\ x &= -apq(p+q) \end{aligned}$$

Hence

$$R\left(-apq(p+q), a[p^2+pq+q^2+2]\right)$$

as required.

Q4.c.ii

Substituting the point $(0, a)$ into the equation $y = \frac{1}{2}(p+q)x - apq$ we obtain, $a = \frac{1}{2}(p+q) \times 0 - apq$ and so $a = -apq$ and cancelling we find $pq = -1$ as required.

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Q4.c.iii We start with the x and y coordinates of R and manipulate them using $pq = -1$ until we have eliminated p, q and are left only with an equation involving x, y.

$$\begin{aligned} x &= a(p+q) \\ y &= a(p^2+pq+q^2+2) \\ \therefore \quad y &= a(p^2+-1+q^2+2) = a(p^2+q^2+1) \\ y/a &= p^2+q^2+1 \\ &= (p+q)^2 - 2pq + 1 \\ &= (x/a)^2 - 2(-1) + 1 \\ y/a &= x^2/a^2 + 3 \\ x^2 &= ay - 3a^2 \\ x^2 &= a(y-3a) \end{aligned}$$

Therfore the equation of the locus of R is $\mathbf{x^2 = a(y - 3a)}$ which is a parabola with vertex $(0, 3a)$.

Q4.d

Step 1: When $n = 2$, LHS$= 4^2 - 1 - 7.2 = 16 - 1 - 14 = 1 > 0$ and so it's true when $n = 2$.

Step 2: Assume it's true when $n = k$, so assume

$$4^k - 1 - 7k > 0 \tag{1}$$

Step 3: Prove it true for $n = k + 1$, so prove that $4^{k+1} - 1 - 7(k+1) > 0$.

Start,

$$\begin{aligned} 4^{k+1} - 1 - 7(k+1) &> 4(1+7k) - 1 - 7k - 7 \text{ using (1)} \\ &= 4 + 28k - 1 - 7k - 7 \\ &= 21k - 4 > 0 \text{ since } k \geq 2 \end{aligned}$$

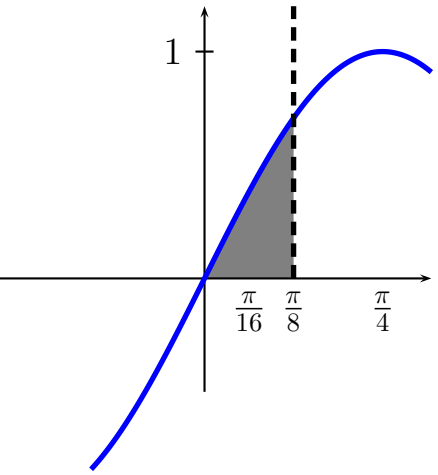
Hence it's true when $n = k + 1$.

Step 4: We showed that the statement was true when $k = 2$. We also showed that the statement is true for $n = k + 1$ whenever the statement is true $n = k$. Hence the statement is true for all integers $n \geq 2$ as required.

Question 5

Q5.a

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$$\begin{aligned} V &= \pi \int_a^b y^2\,dx = \pi \int_0^{\pi/8} \sin^2 2x\,dx \\ &= \frac{\pi}{2} \int_0^{\pi/8} (1-\cos 4x)\,dx \\ &= \frac{\pi^{\mathbf{2}}}{\mathbf{16}} - \frac{\pi}{\mathbf{8}} \end{aligned}$$

Q5.b

Q5.b.i

We are given that $\angle PDC = \angle PAB = 90^o$ and so opposite angles of the quadrilateral $DPAE$ add up to 180^o then it must be a cyclic quadrilateral, as required.

Q5.b.ii

$\angle APE = \angle ADE$ (angles on same chord AE on circle through $DPAE$

$\angle ADE = \angle ABC$ (angles on same chord AC on circle through $ADBC$

$\therefore \angle APE = \angle ABC$ as required.

Q5.b.iii Since $\angle APE = \angle ABC$ then $APBQ$ is a cyclic quadrilateral.

$\angle PAB = \angle PBQ$ (equal angles on chord PB)

$\angle PQB = 90^o$ ()since $\angle PAB = 90^o$ is given)
Hence $PQ \perp BC$

Q5.c

Q5.c.i

$x = 5 + \sqrt{3}\sin 3t - \cos 3t$

$\sqrt{3}\sin 3t - \cos 3t = R\sin(3t - \alpha)$

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Expanding the right side,

$\sqrt{3}\sin 3t - \cos 3t = (R\cos\alpha)\sin 3t - (R\sin\alpha)\cos 3t$

Choose R, α so that

$R\cos\alpha = \sqrt{3}, \quad R\sin\alpha = 1$

$R = \sqrt{3+1} = 2$

$\tan\alpha = \dfrac{1}{\sqrt{3}}$

$\alpha = \tan^{-1}\dfrac{1}{\sqrt{3}} = \dfrac{\pi}{6}$

So we may write $\sqrt{3}\sin 3t - \cos 3t = 2\sin(3t - \dfrac{\pi}{6})$

Q5.c.ii

Write the equation as, $x - 5 = 2\sin(3t - \frac{\pi}{6})$.
Hence the amplitude is 2 and the centre of motion is $x = 5$.

Q5.c.iii

For SHM the maximum speed occurs at the centre of motion and so we solve
$\sin(3t - \dfrac{\pi}{6}) = 0$
$3t - \pi/6 = 0, \pi, 2\pi, \cdots$

For the first positive solution, $3t = 0 + \pi/6 = \pi/6$ and $\mathbf{t = \dfrac{\pi}{18}}$

Question 6

Q6.a

Recall that in binomial probability,

The Probability of r successess from n trials equals $\binom{n}{r} p^r q^{n-r}$, where the probability of a success is p and $q = 1 - p$ is the probability of a fail.

Q6.a.i

The number of trials is $n = 5$, the probability of success is $p = \dfrac{2}{3}$, and $r = 3, 4, 5$ so

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The probability of Megan scoring one point	$=$	$\binom{5}{3}p^3q^2 + \binom{5}{4}p^4q + \binom{5}{5}p^5$
	$=$	$10\left(\frac{2}{3}\right)^3\left(\frac{1}{3}\right)^2 + 5\left(\frac{2}{3}\right)^4\frac{1}{3} + \left(\frac{2}{3}\right)^5$
	$=$	$\frac{64}{81} = 0.7901$

as required.

Q6.a.ii

$\left(\frac{64}{81}\right)^{18} = \mathbf{0.01439}$ or about $\mathbf{1.44}\%$.

Q6.a.iii $p = 0.7901$, $q = 1 - p = 0.2099$

Number of trial is $n = 18$. The probability of at most 16 successess is equivalent to finding the one minus the probability of 17 or 18 successes.

The probability of Megan scoring at most 16 points	$=$	$1 - \binom{18}{17}(0.7901)^{17}(0.2099) - \binom{18}{18}(0.7901)^{18}$
	$=$	$0.9167566 = \mathbf{0.92}$

Q6.b

Q6.b.i

$$\begin{aligned} x &= 200t \\ y &= -4.9t^2 + 200t + 5000 \\ \dot{x} &= 200 \\ \dot{y} &= -9.8t + 200 \end{aligned}$$

Maximum height occurs when $\dot{y} = 0$, and so $t = \frac{200}{9.8} = \mathbf{20.408}$ s. The maximu height is $y = -4.9^2(200/9.8) + 200.200/9.8 + 5000 = \mathbf{7040.82}$m

Q6.b.ii If the angle of descent of the rocket is ϕ then $\tan\phi = \frac{\dot{y}}{\dot{x}}$ We require that

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$$\begin{array}{rcccl}
-60^o & \leq & \phi & \leq & -45^o \\
\tan(-60) & \leq & \tan\phi & \leq & \tan(-45) \\
-\sqrt{3} & \leq & \tan\phi & \leq & -1 \\
-\sqrt{3} & \leq & \dfrac{\dot{y}}{\dot{x}} & \leq & -1 \\
-\sqrt{3} & \leq & \dfrac{200-9.8t}{200} & \leq & -1 \\
\sqrt{3} & \geq & \dfrac{9.8t}{200}-1 & \geq & 1 \\
2 & \leq & \dfrac{9.8t}{200} & \leq & \sqrt{3}+1 \\
\dfrac{400}{9.8} & \leq & t & \leq & \dfrac{200(\sqrt{3}+1)}{9.8} \\
\mathbf{40.816}\text{ s} & \leq & t & \leq & \mathbf{55.756}\text{ s}
\end{array}$$

Q6.b.iii

$v^2 = \dot{x}^2 + \dot{y}^2 = 200^2 + (200-9.8t)^2$

So here we have,

$$\begin{array}{rcl}
350^2 & \geq & 200^2 + (200-9.8t)^2 \\
350^2 & \geq & 2\times 200^2 - 2\times 9.8\times 200t + 9.8^2t^2 \\
0 & \geq & 96.04t^2 - 3920t - 42500 \\
t & = & \dfrac{3920 \pm \sqrt{3920^2 + 4\times 42500\times 96.04}}{2\times 96.04} \\
t & \leq & 49.717
\end{array}$$

So the pilot can eject safely at the very latest when $\mathbf{t = 49.7}s$.

Question 7

Q7.a

Q7.a.i The triangle from P to the diameter is isosceles so $\tan\dfrac{\alpha}{2} = \dfrac{PT}{r}$ so $r = PT\,\tan\frac{\alpha}{2}$.

$PT = \sqrt{2^2 + 0.45^2} = 2.05$

So $r = 2.05\tan\dfrac{0.1}{2} = 0.1025855 = 103$ m.

Q7.a.ii $\dfrac{dr}{dt} = \dfrac{2.05}{2}.\dfrac{d\alpha}{dt}\sec^2\theta$

$\dfrac{dr}{dt} = \dfrac{2.05}{2}.\times 0.02\times \sec^2\dfrac{0.1}{2} = 0.02055 = 20.6$ m/hr

Q7.b $f(x) = Ax^3 - Ax + 1, A > 0$

$f'(x) = 3Ax^2 - A$

$f''(x) = 6Ax$

Q7.b.i For stationary points $f'(x) = 0$ so $3Ax^2 - A = 0$ and $x^2 = 1/3$ and so $x = \pm\dfrac{1}{\sqrt{3}}$

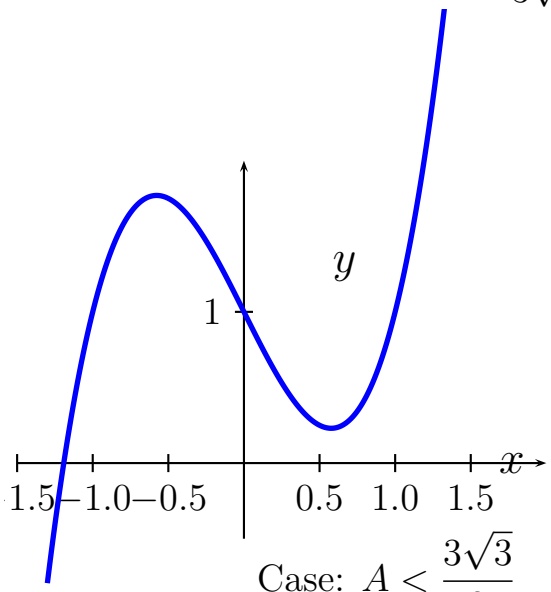
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and the two stationary points are

$A\left(-\frac{1}{\sqrt{3}}, 1+\frac{2A}{3\sqrt{3}}\right)$ is a Maximum since $f''(-\frac{1}{\sqrt{3}}) < 0$.

$B\left(\frac{1}{\sqrt{3}}, 1-\frac{2A}{3\sqrt{3}}\right)$ is a Minimum since $f''(\frac{1}{\sqrt{3}}) > 0$.

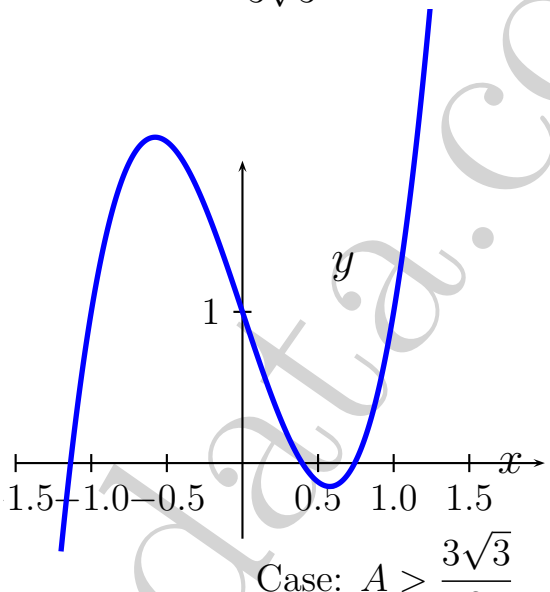
Q7.b.ii When $A < \frac{3\sqrt{3}}{2}$ then $\frac{2A}{3\sqrt{3}} < 1$ and so $1 - \frac{2A}{3\sqrt{3}}$.



Case: $A < \frac{3\sqrt{3}}{2}$

$y = Ax^3 - Ax + 1$

$A = 3$



Case: $A > \frac{3\sqrt{3}}{2}$

$y = Ax^3 - Ax + 1$

$A = 3$

This means that the Minimum point found above will lie above the x-axis when $A < \frac{3\sqrt{3}}{2}$ (example $A = 2$) and so in this case there is exactly one zero.

When $A > \frac{3\sqrt{3}}{2}$ then the MInimum point lies below the x-axis giving rise to two more roots.

Q7.b.iii Since $f(-1) = 1$ it is evident from the graph that the only zero occurs for $x < -1$ and so there are no zeros in the interval $-1 \le x \le 1$ when $0 < A < \frac{3\sqrt{3}}{2}$.

Q7.b.iv $g(\theta) = 2\cos\theta + \tan\theta$, $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$.

$g'(\theta) = \sec^2\theta - 2\sin\theta = 0$ for stationary points.

$$\begin{aligned}
\sec^2\theta &= 2\sin\theta \\
\frac{1}{\cos^2\theta} &= 2\sin\theta \\
2\sin\theta\cos^2\theta - 1 &= 0 \\
2\sin\theta(1-\sin^2\theta) - 1 &= 0 \\
2\sin\theta - 2\sin^2\theta - 1 &= 0 \\
2\sin^2\theta - 2\sin\theta + 1 &= 0
\end{aligned}$$

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Keeping in mind that $-1 \leq \sin\theta \leq 1$, then by part (iv) this cannot have a root. This corresponds to the case where $A = 2$ and the condition on A is satisfied as $2 < \dfrac{3\sqrt{3}}{2}$.

Q7.b.v First, $g'(0) = 1 - 2\sin 0 = 1 > 0$ so that $g'(\theta) > 0$ for $-\dfrac{\pi}{2} < \theta < \dfrac{\pi}{2}$.
Hence an inverse function exists since $g(\theta)$ satisfies the horizontal line test.

The End

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